Lecture 11

Review

- We are classifying irreducible $\mathbb{C}\Sigma_n$ modules, we know the # of them is the # of partitions of n
- Using that $C_{\mathbb{C}\Sigma_n}(\Sigma_{n-1})$ is commutative we showed $Res_{\Sigma_{n-1}} V$ is multiplicity free for any simple Σ_n-module V.
- Defined Jucys-Murphy elements $L_1 = 0, L_2 = (12), L_3 = (13)+(23), \ldots, L_n = (1n)+\cdots+(n-1\,n)$

Def The Branching graph $\mathcal{B}$ has vertices the irreducible $\mathbb{C}\Sigma_n$ modules $\forall n$ and edges $W \to V$ if $Res^{\Sigma_n}_{\Sigma_{n-1}} V \cong W \oplus \_\_$

Def For V an irreducible $\mathbb{C}\Sigma_n$ module, we saw V has a Gelfand-Zetlin basis, unique up to scalars, with each basis vector corresponding to a path $W_0 \to W_1 \to \cdots \to W_n = V$ in $\mathcal{B}$

Rmk For V_T corr to this path, $\mathbb{C}\Sigma_k \cdot V_T = W_k$

GZ subalgebra

1. Write $\mathbb{C}\Sigma_n \cong \bigoplus_{V \text{ irred}} End_{\mathbb{C}}(V)$ This is canonical.

2. Choose the GZ basis of each V to get $\mathbb{C}\Sigma_n \cong \bigoplus_V M_{\dim V}(\mathbb{C})$ (*)

Def The GZ subalgebra of $\mathbb{C}\Sigma_n$ corresponds to matrices diagonal in each coordinate in (*). This is clearly a maximal commutative, semisimple algebra, call it A_n

Thm $A_n = \langle L_1, L_2, \ldots, L_n \rangle$

Def Suppose V is an irreducible $\mathbb{C}\Sigma_n$-module, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n - \{\vec{0}\}$. Say λ is a <u>weight</u> of V if $\exists v \neq 0$ $L_i v = \lambda_i v$ $\forall i$. The set of such v is the λ-weight space V_λ. Note: V is a $\oplus$ of weight spaces

Theorem

1. Elements of a GZ basis are weight vectors
2. In an irreducible $\mathbb{C}\Sigma_n$ module, all weight spaces are 1-dim.
3. If λ is a weight of an irreducible V, it is not a weight of any other irreducible $V' \neq V$. (Write $V(\lambda) = V$)
4. There is a 1-1 correspondence between all weights for all Σ_n and paths in $\mathcal{B}$ from W_0.

Proof
1. Since $L_i \in A$ then matrix of L_i is diagonal in GZ basis.
2. If two different v_T had same action of all L_i this would contradict A being <u>all</u> diagonal matrices
3. Same as 2
4. We know weights $\leftrightarrow$ GZ basis vectors $\leftrightarrow$ paths //

Remark Suppose $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ a weight of V. Then $(\lambda_1, \ldots, \lambda_{n-1})$ is a weight of Res V. To understand $\mathcal{B}$ we need:

1. Which weights in $\mathbb{C}^n$ actually occur, call this $W(n)$,
2. For $\lambda, \mu \in W(n)$, when are $\lambda\ \mu$ in same irreducible?

Notation $\lambda \in W(n)$ then $V(\lambda)$ is the irreducible with λ as a weight.

$V_\lambda = V_{T_\lambda}$ is the corresponding GZ basis vector.

Degenerate Affine Hecke Algebra

Def $\mathcal{H}_2 = \langle s, x, y \mid xy = yx,\ s^2 = 1,\ sx = ys - 1 \rangle$ (thus $xs = sy - 1$)

Irreducible $\mathcal{H}_2$-modules : Fix $a, b \in \mathbb{C}$, construct $L(a,b)$:

Case 1 $a = b - 1$ $\quad x \to (a)$ $y \to (b)$ $s \to (1)$

Case 2 $a = b + 1$ $\quad x \to (a)$ $y \to (b)$ $s \to (-1)$

Case 3 $a \neq b \pm 1$ $\quad x \to \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}$ $y \to \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ $s \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

* Subcase $a = b$ then x, y do not act semisimple

* Subcase $a \neq b, b \pm 1$ New basis

$$x \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad y \to \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \quad s \to \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}$$

Thm Every irreducible $\mathcal{H}_2$-module is $\cong$ to some $L(a,b)$.
If $a \neq b \pm 1$ then $L(a,b) \cong L(b,a)$, and no further $\cong$ exist.

"**Proof**" Choose v a simultaneous eigenvector for x, y w/ evalues a, b. Consider $\langle v, sv \rangle$

$$ysv = (sx + 1)v = sxv + v = a\,sv + v \quad \text{etc.} \quad //$$

Exercise $\langle s_i, L_i, L_{i+1} \rangle$ satisfy relations of $\mathcal{H}_2$. Thus $\exists$ a surjection

$$\mathcal{H}_2 \twoheadrightarrow B_i := \langle s_i, L_i, L_{i+1} \rangle$$

Classifying $W(n)$

Thm Suppose $\lambda \in W(n)$, $V = V(\lambda)$, $\mu = s_i\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_n)$

1. $\lambda_i \neq \lambda_{i+1}$
2. If $\lambda_i = \lambda_i \pm 1$ then $s_i V_\lambda = \pm V_\lambda$ and μ is not a weight of V.
3. If $\lambda_i \neq \lambda_i \pm 1$ then μ is a weight of V with nonzero wt vector

$$w := \left(s_i - \frac{1}{\lambda_{i+1} - \lambda_i}\right) v_{\lambda},$$

$\langle v_\lambda, w \rangle$ is 2-dim B_i module $\cong$ subcase $L(\lambda_i, \lambda_{i+1})$

Proof We know $\mathbb{C}\Sigma_{i+1} \cdot v_\lambda \cong V(\lambda_1, \lambda_2, \ldots, \lambda_{i+1})$. Conside

$$M = \operatorname{Hom}_{\Sigma_{i-1}}\left(V(\lambda_1, \lambda_2, \ldots, \lambda_{i-1}), \operatorname{Res} V(\lambda_1, \ldots, \lambda_{i+1})\right)$$

as a $Z_{i-1,2} = \langle B_i, Z_{i-1} \rangle$ module, it is irreducible
B_i and Z_{i-1} commutes at Z_{i-1} acts as scalars by Schur so
M ~~$Z_{i-1,2}$~~ is an irreducible B_i-module

Check M is irreducible H_2-module w/ e-values λ_i, λ_{i+1}.

1. $Z_{i-1,2}$ semisimple but x, y not semisimple in $L(a,a)$ so $\lambda_i = \lambda_{i+1}$ impossible

Rest uses H_2-modules

COR Suppose $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. If $\lambda_i = \lambda_{i+2} = \lambda_{i+1} \pm 1$
then $\lambda \notin W(n)$

Proof $s_i v_\lambda = \pm v_\lambda$, $s_{i+1} v_\lambda = \mp v_\lambda$ contradicts braid

Final Classification of $W(n)$

Thm. Let $\lambda \in \mathbb{C}^n$. Then $\lambda \in W(n) \Leftrightarrow$

1. $\lambda_1 = 0$
2. $\{\lambda_i - 1, \lambda_i + 1\} \cap \{\lambda_1, \lambda_2, .., \lambda_{i-1}\} \neq \emptyset \quad \forall 1 < i \leq n$
3. If $\lambda_i = \lambda_j = a$ some $i < j$ then $\{a-1, a+1\} \subseteq \{\lambda_{i+1}, .., \lambda_{j-1}\}$

Proof

1. Obvious, $L_1 = 0$
2. Suppose 2 fails $\quad \lambda_1, \lambda_2, .., \lambda_{i-1}, \lambda_i$ (No $\lambda_i - 1$ or $\lambda_i + 1$)

   Apply 3 repeatedly to get $\lambda_i \lambda_1, \ldots$
   If $\lambda_i \neq 0$ contradicts 1.
   If $\lambda_i \neq 0$ swap again $\nLeftarrow$ $\lambda_1 = 0$

3. ~~suppose not~~ Pick i, j minimizing $j - i$ where it fails

   $\lambda = (, .., a, \uparrow \quad \uparrow a)$

   must be $a \pm 1$ else can swap

   ~~WLOG say~~ $a+$

   So say

   $a, a \pm 1, .., a \pm 1 \, a$

   must have $(a \pm 1) \pm 1$ //