# Lecture 10

Let $\Sigma_d$ denote the symmetric group on $d$ letters.

Goal! • Construct all irreducible $\mathbb{C}\Sigma_n$ modules, including actual matrices and dimensions

• Describe branching behavior (i.e. $\mathrm{Res}^{\Sigma_d}_{\Sigma_{d-1}} S$)

• Formula for irreducible characters

• See how partitions & tableau arise naturally.

Approach: Famous 1996 paper of Okounkov-Vershik

## Review

• conjugation, conjugacy classes

• conjugation in $\Sigma_n$

• Description of $Z(kG)$ in terms of class sums

• Presentation of $\Sigma_n$ w/ Coxeter generators

Recall $A$ a semisimple algebra (say $/\mathbb{C}$) then

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_t}(\mathbb{C})$$

A simple $A$-module is $n_i$-dimensional, other summands act as zero.

Prop Let $V$ be an $A$-module. Then action of $A$ gives a map $A \xrightarrow{\rho_V} \mathrm{End}_{\mathbb{C}}(V) \cong M_{\dim(V)}(\mathbb{C})$. Then $V$ is simple

iff $\rho_V$ is onto, i.e. Every linear map $V \to V$ is realized by element of $A$.

## Basic Lemma

Let $B \subset A$ be a subalgebra. Let $C = C_A(B) = \{a \mid ab = ba \ \forall b \in B\}$

Def Let $V \in A$-mod, $W \in B$-mod. Then $Hom_B(W, V_B)$ is a $C$-module via:

$$(cf)(w) = cf(w)$$

Just check $cf$ is still a $B$-module homo.

**Lemma** Let $B \subseteq A$ be semisimple f.d. algebras, $V$ an irreducible $A$-module and $W$ an irred. $B$-module. Then

$$Hom_B(W, V_B) \text{ is an irreducible } C\text{-module}$$

Proof By Wedderburn we can assume WLOG that $A \cong End_k(V) = M_{\dim(V)}(k)$

ETS that $C$ is the full $End_k(Hom_B(W, V_B))$

Write $\quad V_B \cong W^{\oplus t} \oplus X$

Then $End_B(W^{\oplus t}) \subseteq C$ acts on $Hom_B(W, V_B)$ as full alg,

## Symmetric Group Setup

**Def** $L_i = (1i) + (2i) + \dots + (i-1, i) \quad 1 \le i \le n$ so

$L_1 = 0 \quad L_2 = (12) \quad L_3 = (13) + (23) \quad L_4 = (14) + (24) + (34)$

called Jucys-Murphy elements

**Lemma** $L_t$ commutes with $\Sigma_{t-1} = \Sigma_{\{1,2,\dots,t-1\}}$

In particular the $\{L_i\}$ commute with each other.

**Notation** $\Sigma_m < \Sigma_n$ means $\Sigma_{\{1,2,3,\dots,m\}}$

$\Sigma_m' < \Sigma_n$ means $\Sigma_{\{n-m+1,\dots,n\}}$

$$Z_n = Z(k\Sigma_n) \qquad Z_{n,m} = (k\Sigma_{n+m})^{\Sigma_n} = C_{k\Sigma_{n+m}}(k\Sigma_n)$$

**Prop** $Z_{n,m}$ is spanned by class sums corresponding to $\Sigma_n$ conjugacy classes in $\Sigma_{n+m}$.

**Rmk** Think of as cycle shapes w/ fixed spots for $n+1, n+2, \dots, n+m$

**Ex** $n=8$ $m=4$ $\quad (* \, 9 * 12 * *)(10 * *)(* * 11)$

sum over all perms w/ $1 \to 8$ replacing $*$'s

**Thm** (Olshanskii) The algebra $Z_{n,m}$ is generated by $S_m'$, $Z_n$ and $L_{n+1}, L_{n+2}, \ldots, L_{n+m}$.

**Proof** All these are obviously in $Z_{n,m}$, so generate $A \subseteq Z_{n,m}$.

**Def** $Z^i_{n,m}$ = span of class sums corr to cycle shapes moving $\leq i$ elts.

We prove $Z^i_{n,m} \subseteq A$ by induction on $i$, thus $Z_{n,m} \subseteq A$.
$i = 0, 1$ clear.

**Example**

$$z \in Z^{12}_{11,4} \leftrightarrow (*****)(**)(*)(*)(12 * 13\ 14 *)(15)$$

Let $c \in Z_{11} \leftrightarrow (*****)(**)$

$$x = (12,13)L_{12}(13,14)\big(L_{14} - (12,14) - (13,14)\big) \in A.$$

Check $xc - z \in Z^{11}_{11,4}$ so $z \in \langle A, Z^{11}_{11,4} \rangle$

**Rmk** $L_{12}$ = class sum $\leftrightarrow (*, 12)$
$L_{14} - (12,14) - (13,14)$ = class sum $\leftrightarrow (*, 14)$ //

**Thm** Let $V$ be an irreducible $\mathbb{C}\Sigma_n$-module. Then $V_{\Sigma_{n-1}}$ is multiplicity free

**Proof** Let $B = \mathbb{C}\Sigma_{n-1}$, $A = \mathbb{C}\Sigma_n$. By Thm above $C_{\mathbb{C}\Sigma_n}(\mathbb{C}\Sigma_{n-1}) = Z_{n-1,1}$ is spanned by $Z_n$, $L_{n+1}$, and so is abelian

Thus $\mathrm{Hom}_B(W, V_B)$ must be one-dimensional! //

## Consequence of Multiplicity Free Branching

**Def** The <u>branching graph</u> has as vertices $\cong$classes of irreducible $\mathbb{C}\Sigma_n$ modules $\forall n \geq 0$ and an edge $W \to V$ from $\Sigma_n$-module $W$ to a $\Sigma_{n+1}$-module $V$ iff $W$ appears in $\mathrm{Res}_{\Sigma_n} V$.

**Prop** Let $V$ be an irred $\mathbb{C}\Sigma_n$ module. Then the decomposition

$$\mathrm{Res}_{\Sigma_{n-1}} V = \bigoplus_{W \to V} W \quad \text{is } \underline{\text{canonical}}!$$

**COR** There is a canonical decomposition

$$\mathrm{Res}_{\Sigma_0}(V) = \bigoplus_T V_T$$

where $V_T$ is one-dimensional and $T$ runs over all paths

(*) $$T = \quad W_0 \to W_1 \to \dots \to W_n = V \quad \text{in } \mathcal{B}.$$

**Def** Choosing a vector $\{v_T \in V_T$, we get a basis of $V$ called Gelfand-Zetlin basis. It is unique up to scalars.

**Prop** 1. Given $T$ as in (*), $\mathbb{C}\Sigma_k \cdot v_T = W_k$

2. Any $\cong$ $\varphi: V \to V'$ takes GZ basis to GZ basis.

3. GZ basis is $\perp$ wrt an $\Sigma_n$-invariant inner product $( \, , \, )$ on $V$.