Lecture 2

Review: k a field, usually alg. closed
A a finite dimensional k-algebra w/ identity.

Ex $A \cong kG$ group algebra of a finite group.

Then an A-module U is a vector space over k with an action of A by linear maps

Excercise Given an n-dimensional A-module U, we get an induced algebra homomorphism

$$\psi : A \to M_{n \times n}(k) \cong End_k(U) \cong Hom_k(U, U)$$

Rmk Given R-modules M, N, $Hom_R(M,N)$ is an abelian group, the set of R-module homomorphisms; $Hom_R(M,M)$ is a ring, the endomorphism ring of M.

$Hom_A(M,M)$ is an algebra, the endomorphism algebra.

Schur's Lemma Let S be a simple A-module.
1. $End_A(S)$ is a division algebra.
2. When ~~A~~ k is alg. closed then $End_A(S)$ ~~is a division algebra~~ $= \{\lambda Id \mid \lambda \in k\}$.

Proof 1. Image and kernel are both submodules so nonzero maps are invertible
2. Chose e-value λ for $f: S \to S$. Look at $f - \lambda I$.

Free module Review

- $A \oplus A \oplus \cdots \oplus A = A^{\oplus n}$ is a free A-module of rank n.
  Let $a_i = (0,0,\ldots,1,0,\ldots,0)$ Then

<u>Universal Property</u> Given any A-module U and $u_1, u_2, \ldots, u_n \in U$, there is a unique A-module homomorphism $\varphi: A^n \to U$ mapping $a_i \to u_i$.

<u>Ex</u> $A = k$, this is a property of basis of vector space

<u>Corollary</u>

1. Every finite dimensional A-module is the homomorphic image of a free module A^n, some n.

2. There are only finitely many simple A-modules up to $\cong$.

<u>Proof</u>

1. Any generating set $\{u_1, \ldots, u_n\}$ of U guarantees map φ is onto, for example a basis of U.

2. Let $s \in S$, $s \neq 0$, ~~$\$$~~ S simple. Then $As = S$ so can choose $n=1$, i.e. $\exists$ a surjection $A \to S$.
   Now A is f.d., so has a composition series, so only finitely many possible S.

<u>Rmk</u> Modules generated by one element are called <u>cyclic</u>. So of course simple $\Rightarrow$ cyclic.

Semisimple Modules

Def An Amodule is <u>semisimple</u> if it is a direct sum of simple modules.

Ex $A = T_2(k)$ natural module is not semisimple.

Lemma Suppose $U \cong S_1 \oplus S_2 \oplus \cdots \oplus S_n$, S_i are simple and suppose $V \leq U$. Then $\exists$ a subset $I \subseteq \{1, 2, \ldots, n\}$ such that

$$U \cong V \oplus \bigoplus_{i \in I} S_i =: V \oplus S_I$$

Proof Choose I maximal so that $S_I \cap V = 0$. Then

$$S_I + V \cong S_I \oplus V, \quad \text{want } S_I + V = U.$$

Suppose not, $\exists\ S_i$ w/ $S_i \cap (S_I + V) = 0$ so
$$S_i + S_I + V \cong S_i \oplus S_I \oplus V, \not\times.$$

Prop Let U be an A-module. TFAE
1. U is a $\oplus$ of simple modules
2. Every submodule of U is a direct summand

Exercise: Check that the Lemma implies submodules and quotients of semisimple modules are semisimple.

Proof $1 \to 2$ by Lemma. Suppose "2" holds for U. I claim it holds for any U/V. Let $W/V \leq U/V$ so $U = W \oplus X$ by 2.
Check $U/V = W/V \oplus X+V/V$.
Now choose simple submodule S, so $U = S \oplus T$, $T \cong U/S$.
But 2 holds for T so T is a $\oplus$ of simple modules by induction on compos. length

The Radical

<u>Recall</u> Let U be an A-module. The <u>annihilator</u> $\operatorname{ann} U = \{a \in A \mid au = 0 \ \forall u \in U\}$. is a 2-sided ideal.

<u>Def</u> The radical, $\operatorname{rad} A = \bigcap_{\substack{\text{simple} \\ A\text{-modules} \\ S}} \operatorname{ann} S$, so this is a 2-sided ideal, the largest ideal annihilating all semisimple A-modules

<u>Recall</u> An ideal I is <u>nilpotent</u> if $I^n = 0$ for some n.

<u>Thm</u> The radical of A is equal to each of:

1. The smallest submodule of ${}_A A$ with semisimple quotient.
2. The $\bigcap$ of all maximal submodules of ${}_A A$ (a.k.a. max left ideals of A)
3. The largest nilpotent ideal.

<u>Def</u> An algebra A is <u>semisimple</u> if $\operatorname{rad} A = 0$.

<u>COR</u> TFAE

1. A is semisimple
2. ${}_A A$ is semisimple
3. Every A-module is semisimple.

<u>Exercise</u> $\operatorname{rad}(A/\operatorname{rad} A)$ is semisimple, i.e. $A/\operatorname{rad} A$ is a semisimple algebra.

Proof of Thm

First note $I^n = J^n = 0 \Rightarrow (I+J)^{m+n} = 0$ so sum of all nilp ideals is nilpotent, so (3) exists, call it N.

If S is simple then NS is a submodule, 0 or S. But $N^n = 0 \Rightarrow N^n S = 0 \Rightarrow NS = 0$.
Thus $N \subseteq \mathrm{rad}\, A$.

Let $0 = A_0 \subset A_1 \subset \ldots \subset A_t = A$ be a comp series.

$$\mathrm{rad}\, A(A_i/A_{i-1}) = 0 \Rightarrow \mathrm{rad}\, A(A_i) \subseteq A_{i-1}$$
$$\Rightarrow (\mathrm{rad}\, A)^t A = 0 \Rightarrow (\mathrm{rad}\, A)^t = 0 \Rightarrow \mathrm{rad}\, A \text{ nilp}$$
$$\Rightarrow \mathrm{rad}\, A \subseteq N.$$

Thus $N = \mathrm{rad}\, A$.

Suppose $M_1, \ldots, M_r$ are maximal submodules so A/M_i is simple so $A/M_1 \oplus \ldots \oplus A/M_r$ semisimple. Let $I = M_1 \cap \ldots \cap M_r$

$$A/M_1 \cap \ldots \cap M_r \hookrightarrow A/M_1 \oplus \ldots \oplus A/M_r \text{ so } A/M_1 \cap \ldots \cap M_r \text{ is s.s}$$

Thus A/I is semisimple. But if A/M is semisimple, say $A/M = L_1/M + \ldots + L_s/M$, build max submodule.
Thus $M \supseteq I$. Thus $(1) = (2)$

Finally Let $a \in \mathrm{rad}\, A$, M a max submodule. Then $aA \subseteq M$ so $a \cdot 1 \in M \Rightarrow a \in M$. Thus $\mathrm{rad}\, A \subseteq I$.

Suppose $\mathrm{rad}\, A \subset I$. Choose S simple so $IS \neq 0$. Choose s so $Is \neq 0$.
Thus $Is = S$ so $\exists\, x \in I$ so $xs = -s$ so $(x+1)s = 0$ so $x+1 \in \mathrm{ann}\, S$
so $x+1 \subseteq$ some max, so $x, x+1 \in I$. $\divideontimes$ EX: $T_n(\mathbb{R})$