- HW
- OFFICE HOURS
- RESERVE

Intro

Group $\Sigma_3 = \langle e, (12), (123), (23), (13), (132) \rangle$

$$G_1 = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \quad G_1 \cong \Sigma_3 \leq GL_3(\mathbb{R})$$

$$G_2 = \left\langle \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \quad G_2 \cong \Sigma_3$$

two different <u>representations</u> of Σ_3

- Rep theory studies how to "represent" a group as matrices.
- Groups acting on vector spaces

Variations

Group: finite, discrete, algebraic
Field: char 0 vs char p, alg closed?

More general: represent fin. dim. algs

Plan
- Alperin for a while (I–IV)
- Sym groups, reps of sym grps,
- Current areas of research

Lecture 1

Review

Def R a ring (w/ identity). A (left) R-module is an abelian group M together with an action $R \times M \to M$, $(r,m) \to rm$ such that

- $(r_1+r_2)m = r_1m + r_2m$
- $r(m_1+m_2) = rm_1 + rm_2$
- $(r_1r_2)m = r_1(r_2m)$
- $1m = m$

$\forall\, r, r_i \in R,\ m, m_i \in M$

similarly for right R-modules.

Examples

1. Left regular module ${}_RR$ $(M = R)$
2. Given R-modules $\{M_i \mid i \in I\}$ then $\bigoplus_{i\in I} M_i$ is an R-module.
3. Any left ideal $I \subset R$, then I is a submodule.
4. Cat of $\mathbb{Z}$-modules equiv cat of abelian groups.

Def A <u>submodule</u> is a subgroup $N \leq M$ such that $rn \in N$ $\forall r \in R, n \in N$.

A module is <u>simple</u> (aka irreducible) if the only submodules are zero and the entire module. INDECOMPOSABLE

Given a submodule $N \leq M$, the quotient group M/N is an R-module via $r(m+N) = rm + N$ $(r\bar{m} = \overline{rm})$, just check well-defined!

Thus no such thing as a "normal submodule".

* MODULE HOMOMORPHISMS, ISOMORPHISMS

Def A <u>composition series</u> for an R-module M is a series of submodules
$$0=M_0 \subset M_1 \subset M_2 \cdots \subset M_n = M$$ such that M_i/M_{i-1} is simple.

A module satisfies the ACC if every ascending chain of submodules stops, and is called Noetherian. Similar D.C.C.

<u>Jordan-Hölder Theorem</u> Given any two series
$$0=M_0 \subseteq \cdots \subseteq M_r = M$$
$$0=M_0' \subseteq \cdots \subseteq M_s' = M$$ we may refine them to

$$0=L_0 \subseteq \cdots \subseteq L_n = M$$
$$0=L_0' \subseteq \cdots \subseteq L_n' = M$$ so that $\{L_i/L_{i-1}\}$ and $\{L_i'/L_{i-1}'\}$ are permutations up to $\cong$. Thus TFAE

1. M has a composition series
2. Every series can be refined to a composition series
3. M satisfies ACC and DCC.

Def The length of a composition series is the <u>composition length</u>.
M is uniserial if it has a unique composition series.

Ex $M = S_1 \oplus S_2$ is not!

Algebras

Rmk Our rings usually have additional vector space structure

Def k a field. A k-algebra is a vector space$/k$ which is also a ring, and operations are compatible, e.g. $\lambda(ab)=(\lambda a)b=a(\lambda b)$ $\forall a,b\in A$ $\lambda\in k$.

Rmk 1. Think of as a vector space w/ multiplication or as a ring w/ scalars
2. Called a finite-dimensional algebra if f.d. as a vector space
3. Usually assume identity, so $\{\lambda\cdot 1\}$ is a copy of the field inside.

Examples

1. Field k is a 1-dim k-algebra
2. Polynomial algebras $k[x_1,x_2,\dots,x_n]$

> commutative

3. Matrix algebras $M_n(k)=\{n\times n \text{ matrices, entries in } k\}$
4. $T_n(k)=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\subseteq M_n(k)\right\}$ subalgebra. (weaker than ideal)
5. Group algebras

   G a group, kG is a vector space w/ G as a basis
   Elements are finite linear combinations of group elements

   - Multiplication is via distributive law
   - kG has a distinguished basis, namely G
   - Other elements are not necessarily invertible
   - Occasionally "group rings" like $\mathbb{Z}G$, RG are useful.