Review: G a finite group.
$\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$

- $e = z_1 + z_2 + \cdots + z_r$ central primitive idempotents
- $S_1, \ldots, S_r$ simple $\mathbb{C}G$-modules, $S_i \cong$ col vectors for $M_{n_i}(\mathbb{C})$
  $\dim S_i = n_i$, $\quad z_i s = z_i \; \forall s \in S_i$.

- Let $\rho_1, \ldots, \rho_r$ be reps, $\chi_1, \chi_2, \ldots, \chi_r$ corr. irreducible characters, $\chi_i(1) = n_i$

Thm: Let M be a $\mathbb{C}G$-module. Then $\chi_M = \sum_{i=1}^{r} a_i \chi_i$ with $a_i \geq 0$ unique so $M \cong S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$. Further $M \cong N$ iff $\chi_M = \chi_N$.

Recall: A *class function* is $f: \mathbb{C}G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g) \; \forall x, g \in G$.

Prop. The set of class functions is a vector space $/\mathbb{C}$ with basis $\{\chi_i\}_{i=1}^{r}$

Basic Properties

Let ρ be a representation w/ character $\chi_\rho = \chi$

1. $\rho(g)$ is diagonalizable.
2. $\chi(g)$ is a sum of n^{th} roots of unity, where $n = \text{order}(g)$. In particular $\chi(g)$ is an algebraic integer.
3. $\rho^*(g) = (\rho(g)^{-1})^{tr}$ is a representation, called contragredient dual, and $\chi_{\rho^*}(g) = \overline{\chi(g)} = \chi(g^{-1})$

4. $|\chi(g)| \leq \chi(1)$ with equality iff $\rho(g) = \lambda I_d$

5. $\chi(g) = \chi(1)$ iff $g \in \ker \rho$

Proof - -

Regular rep..

Lemma Let π be the character of regular representation. Then:

1. $\pi = \sum_{i=1}^{r} n_i \chi_i = \sum_{i=1}^{r} \chi_i(1)\chi_i$ so $\pi(g) = \sum_{i=1}^{r} \chi_i(1)\chi_i(g)$

2. $\pi(g) = \begin{cases} |G| & g=1 \\ 0 & \text{else} \end{cases}$

Now we can compute C.O.B matrix from conj. class sums to $\{z_i\}$.

Prop $z_i = \frac{1}{|G|} \sum_{g \in G} \chi_i(1)\chi_i(g^{-1}) g$

Proof

Let $z_i = \sum_{g \in G} a_g g$. Then:

$$\pi(z_i g^{-1}) = a_g |G| = \sum_{j=1}^{r} \chi_j(1)\chi_j(z_i g^{-1})$$

But $\rho_j(z_i g^{-1}) = \rho_j(z_i)\rho_j(g^{-1}) = \delta_{ij}\rho_j(g^{-1})$. Thus

$$a_g |G| = \chi_i(1)\chi_i(g^{-1}) \text{ so } a_g = \frac{1}{|G|}\chi_i(1)\chi_i(g^{-1}) \quad //$$

Def The <u>character table</u> of G has rows indexed by irreducible characters, columns by conjugacy classes, and entries are character values

- Thus it is a square matrix
- Traditionally 1^{st} row is triv. char, 1^{st} col is identity

Ex

$G = S_3$

	$\{e\}$	(12)	(123)
triv $= \chi_1$	1	1	1
sgn $= \chi_2$	1	-1	1
χ_3	2	0	-1

Notice 1^{st} col is character degrees

Thm (Orthogonality Relations) Let $g_i \in \mathcal{C}_i$ where $\{\mathcal{C}_i\}_{i=1}^r$ are conj classes

1. (Row $\perp$) $\quad \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}$

2. (Col $\perp$) $\quad \sum_{k=1}^{r} \chi_k(g_i)\chi_k(g_j) = \delta_{ij}\,|C_G(g_i)| = \delta_{ij}\frac{|G|}{|\mathcal{C}_i|}$

Rmk Later: $\langle \chi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\psi(g)}$ is inner product on class functions

$\{\chi_i\}$ is orthonormal basis

Ex $G = D_8$ $\quad G' = Z(G) = \{e, r^2\}$

$G/G' \cong V = \{\cancel{lem}\ eZ, sZ, rZ, srZ\}$

So 4 1-dim reps

	e	r^2	$\{s, sr^2\}$	$\{r, r^3\}$	$\{sr, sr^3\}$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

red entries from $\perp$ relations

Proof of Orthogonality Relations

We know $z_i z_j = \delta_{ij} z_i$. Use formula for z_i:

<u>Row</u>

$$\left(\frac{1}{|G|}\sum_{x\in G} \chi_i(1)\chi_i(x^{-1})\,x\right)\left(\frac{1}{|G|}\sum_{y\in G}\chi_j(1)\,\chi_j(y^{-1})\,y\right)$$
$$= \left(\frac{1}{|G|}\sum_{g\in G}\chi_i(1)\,\chi_i(g^{-1})g\right)\delta_{ij}$$

Compare coef of identity on each side:

<u>LHS</u> $\frac{1}{|G|^2}\sum_{g\in G}\chi_i(1)\chi_i(g^{-1})\chi_j(1)\,\chi_j(g)$

<u>RHS</u> $\frac{1}{|G|}\delta_{ij}\;\chi_i(1)^2$

so $\frac{1}{|G|}\delta_{ij}\,\chi_i(1)^2 = \frac{1}{|G|^2}\chi_j(1)\,\chi_j(1)\sum_{g\in G}\chi_i(g^{-1})\chi_j(g)$ //

<u>Column</u> Start w/ row $\perp$ and collect terms

$$\frac{1}{|G|}\sum_{k=1}^{r}\chi_i(g_k)\,\overline{\chi_j(g_k)}\,\frac{|G|}{|C_G(g_k)|} = \delta_{ij}$$

$$\sum_{k=1}^{r}\chi_i(g_k)\left(\frac{1}{|C_G(g_k)|}\,\overline{\chi_j(g_k)}\right) = \delta_{ij}$$

Let A = $r\times r$ matrix $a_{ik} = \chi_i(g_k)$ char table
B = $r\times r$ matrix $b_{kj} = \overline{\chi_j(g_k)}/|C_G(g_k)|$ so $AB = I$

Thus $BA = I$ so $\sum_{k=1}^{r} b_{jk}\,a_{ki} = \delta_{ij}$. Thus

$$\sum_{k=1}^{r}|C_G(g_j)|^{-1}\,\overline{\chi_k(g_j)}\,\chi_k(g_i) = \delta_{ij}.$$

$$\sum_{k=1}^{r}\chi_k(g_i)\,\overline{\chi_k(g_j)} = |C_G(g_i)|\,\delta_{ij}$$ //

Recall: 1. Alg integers are roots of monic polynomials in $\mathbb{Z}[x]$, they form a subring of $\mathbb{C}$.

2. $\{\text{Alg integers}\} \cap \mathbb{Q} = \mathbb{Z}$.

Lemma Suppose $x = \sum_{g \in G} a_g g \in \mathbb{C}G$ and $a_g \in \mathbb{Z}$. Suppose $u \in \mathbb{C}G$ and $xu = \lambda u$. Then λ is an alg int.

Proof Let $G = \{g_1, \ldots, g_n\}$. Then $x g_i = \sum_j a_{ji} g_j$ with $a_{ji} \in \mathbb{Z}$. Thus x acts by integer matrix A on $\mathbb{C}G$ and λ is e-value //

Thm. Let C_j be conj class sum and $\omega_{ij} = \frac{|G|}{|C_G(g_j)|} \cdot \frac{\chi_i(g_j)}{\chi_i(1)}$

1. $\rho_i(C_j) = \omega_{ij} I$

2. ω_{ij} is an alg. integer.

Proof 1. Since C_j is central, $\rho_i(C_j) = \lambda C_j$ same λ. But

$$\rho_i(C_j) = \sum_{g \in C_j} \rho_i(g) \text{ so}$$

$$\begin{aligned} tr(\rho_i(C_j)) &= |\text{Class } C_j| \, \chi_i(g_j) \\ &= \frac{|G|}{|C_G(g_j)|} \chi_i(g_j) \end{aligned}$$

Thus $\lambda = \frac{|G|}{|C_G(g_j)|} \cdot \frac{\chi_i(g_j)}{\chi_i(1)}$

2. We just showed C_i acts by scalar ω_{ij} on $S_j \subseteq \mathbb{C}G$, so that scalar is an alg int, by Lemma. //

Thm Let χ be irreducible. Then $\chi(1) \mid |G|$.

Proof
$$\sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|} \frac{\chi(g_i)}{\chi(1)} \overline{\chi(g_i)} = \sum_i \omega_{si} \overline{\chi(g_i)} \text{ is alg int.}$$

But this is $\frac{|G|}{\chi(1)}$ by col orthogonality.

Thus $\frac{|G|}{\chi(1)} \in \mathbb{Q} \cap \{\text{alg int}\} = \mathbb{Z}$. //

Commutator Char

Def Let $g \in G$. Define $\chi^{[,]}(g) = \#\{(x,y) \mid g = xyx^{-1}y^{-1}\}$.

$\chi^{[,]}$ is obviously a class function.

Exercise $\chi^{[,]} = \sum_{i=1}^{r} \frac{|G|}{\chi_i(1)} \chi_i$.

Problem Find a $\mathbb{C}G$-module with character $\chi^{[,]}$, thus proving $\chi(1) \mid |G|$ w/out alg integer method.

Ex. Character table of S_4

- S_3 is homomorphic image S_4/V
- 2 more via $\perp$

$e \to e$

$(12)(34) \to e$

$(12) \to (12)$

$(1234) \to (12)$

$(123) \to (123)$

$\lvert C(a)\rvert$ Class	1	3	6	6	8
rep	1	(12)(34)	(12)	(1234)	(123)
χ_1	1	1	1	1	1
χ_2	1	1	−1	−1	1
χ_3	2	2	0	0	−1
χ_4	3	$a=-1$	1	−1	0
χ_5	3	$b=-1$	−1	1	0

$3a+3b=-6$

$a^2+b^2=1$

Burnside $p^a q^b$ Thm

Suppose $|G| = p^a q^b$ with p, q prime. Then G is solvable.

Proof WLOG $p < q$.

Lemma 1 Suppose α is a sum of n roots of unity and α/n is an alg int. Then $\alpha/n = 0$ or a root of unity.

Proof Let $f(x)$ be minpoly(α/n), w/ constant term c. Let $B_1, \ldots, B_t$ be Galois conjugates of α/n, so $c = \prod B_i$.

Now $\alpha/n = (\lambda_1 + \ldots + \lambda_n)/n$ each λ_i a root of unity, then each B_i is also of this form. Thus $|B_i| \leq 1$ $\forall n$.

Case 1 Some $|B_i| < 1 \Rightarrow c = 0 \Rightarrow \alpha/n = 0$

Case 2 See $|\alpha/n| = 1$ so $\alpha = n\lambda$. //

Lemma 2 Suppose $\left(\chi_i(1), \frac{|G|}{|C_G(g_j)|}\right) = 1$ for some i, j.

Then either $\chi_i(g_j) = 0$ or $\chi_i(g_j) = \lambda \chi_i(1)$ some root of unity λ.

Proof $\exists a, b \in \mathbb{Z}$ s.t. $a\chi_i(1) + b\frac{|G|}{|C_G(g_j)|} = 1$ so

$$a\chi_i(g_j) + \underbrace{b\frac{|G|}{|C_G(g_j)|}\frac{\chi_i(g_j)}{\chi_i(1)}}_{\omega_{ij}} = \frac{\chi_i(g_j)}{\chi_i(1)}$$

Thus $\frac{\chi_i(g_j)}{\chi_i(1)}$ is an alg integer, apply Lemma 1. //

Thm Suppose $g \in G$ and $|\text{Class } g| = p^a > 1$. Then G is not simple.

Proof

Col $\perp$ gives
$$1 + \sum_{i=2}^{r} \chi_i(1)\chi_i(g) = 0.$$

If p divided every $\chi_i(1)$ such that $\chi_i(g) \neq 0$ then we would get
$$1 + \sum p \cdot \frac{\chi_i(1)}{p} \chi_i(g) = 0$$
$$-1 = p \sum \frac{\chi_i(1)}{p} \chi_i(g) \quad \text{so} \quad -1/p \text{ is alg int.} \; //$$

Thus $\exists\, j$ with $\chi_j(g) \neq 0$ and $p \nmid \chi_j(1)$

Thus $\left(\chi_j(1), \frac{|G|}{|C_G(g)|}\right) = 1$ so
$$\chi_j(g) = \lambda \chi_j(1) \text{ by Lemma 2, some nonprincipal char } \chi_j.$$

Thus $\rho_j(g) \in Z(\rho(G))$ so G is not simple. //

Rmk No known group theory proof of this thm!!

COR $p^a q^b$ thm

Proof Let $x \in Z(\text{Syl}_p)$. Then $P \leq C_G(x)$ so
$$|\text{Class } x| = q^c \quad \times$$