

Lecture 2

Review Functor $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ can be covariant or contravariant.

Def \mathcal{F} is full if $\mathcal{F}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ is surjective $\forall A, B$

full subcategory means inclusion functor is full

\mathcal{F} is faithful if $\mathcal{F}: \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ is 1-1 $\forall A, B$

Remark Opposite category reduces study to either contravariant or covariant, although no reason to do this!

Def Suppose $\mathcal{F}, \mathcal{G}: \mathcal{C} \rightarrow \mathcal{D}$ are covariant functors.

A natural transformation from \mathcal{F} to \mathcal{G} is a map η that assigns to each object $A \in \mathcal{C}$ a morphism

$\eta_A \in \text{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{G}A)$ such that $\forall A, B \in \mathcal{C}, \forall f \in \text{Hom}_{\mathcal{C}}(A, B)$,

the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}A & \xrightarrow{\eta_A} & \mathcal{G}A \\ \mathcal{F}f \downarrow & & \downarrow \mathcal{G}f \\ \mathcal{F}B & \xrightarrow{\eta_B} & \mathcal{G}B \end{array}$$

Remark 1. Obvious alteration for natural transt between contravariant functors.

2. Easy exercise that composition of two nat trans is natural trans

3. If all η_A are \cong 's then called a natural \cong .
"naturally isomorphic" is an equiv. relation on functors $\mathcal{C} \rightarrow \mathcal{D}$.

4. Suppose M is a left R -module. Then $R \otimes_R M$ and M are "naturally isomorphic".

This is shorthand for

$R \otimes_R -$ and Id are naturally \cong functors.

Example 1

V a k -vector space, $eV \in V^{**}$ defined by $eV(f) = f(v)$
 $\forall f \in V^*$

Define

$D^2: k\text{-vec space} \rightarrow k\text{-vec space}$

$D^2 V = V^{**}$

$T: V \rightarrow W$

$D^2 T: \begin{matrix} W^{**} & \xrightarrow{\quad} & V^{**} \\ \xrightarrow{\quad} & & \end{matrix} \quad V^{**} \rightarrow W^{**}$
 $eV \rightarrow eV_{T(v)}$

Thm Let $\eta_V: V \rightarrow V^{**}$ be given by $v \rightarrow eV$.

Then η gives a natural isomorphism from the identity functor to the functor D^2 .

Proof

1. \mathcal{T}_V is an ε from $\text{Id } V \rightarrow D^2 V$.

2. Suppose

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \text{ev} \downarrow & & \downarrow \text{ev} \\ V^{**} & \xrightarrow{D^2 T} & W^{**} \end{array}$$
$$\begin{array}{ccc} V & \rightarrow & T_V \\ \downarrow & & \downarrow \\ \text{ev}_V & \rightarrow & \text{ev}_{T_V} \end{array}$$

✓ commutes!

Example 2

Define a functor $\mathcal{G}_n: \text{CRing} \rightarrow \text{Grp}$ by $R \rightarrow \text{GL}_n(R)$

~~$R \rightarrow \text{U}(R)$~~

and $f: R \rightarrow S$ goes to map $\text{GL}_n(R) \rightarrow \text{GL}_n(S)$
act on elements

Define a functor $\mathcal{G}: \text{CRing} \rightarrow \text{Grp}$ by $R \rightarrow \text{U}(R)$

Thm $\det: \text{GL}_n(R) \rightarrow \text{U}(R)$ gives a natural trans
from \mathcal{G}_n to \mathcal{G} .

Isomorphisms & Equivalences

Def Two categories \mathcal{C}, \mathcal{D} are \cong if $\exists \mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$
with $\mathcal{F}\mathcal{G} = \text{Id}_{\mathcal{D}}, \mathcal{G}\mathcal{F} = \text{Id}_{\mathcal{C}}$

Ex

1. $\mathbb{Z}\text{-mod} \cong \text{Ab}$

2. Etc abelian p -groups \cong vs over F_p

Rmk Not useful, requires same # of objects in each \cong class

Def \mathcal{C} and \mathcal{D} are equivalent if \exists functors $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}, \mathcal{G}: \mathcal{D} \rightarrow \mathcal{C}$
such that $\mathcal{G}\mathcal{F}$ is naturally \cong to $\text{Id}_{\mathcal{C}}$
 $\mathcal{F}\mathcal{G}$ " " " " $\text{Id}_{\mathcal{D}}$.

\mathcal{F} (and \mathcal{G}) are called equivalences of categories.

Rmks

1. Equivalence of categories is an equiv. relation

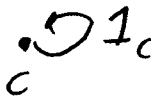
2. Contravariantly equiv vs covariantly equiv.

3. Many important conjectures / thms in rep are about equivalences of categories

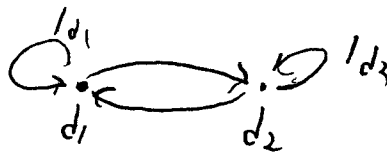
** 4. \mathcal{F} is an equivalence of categories iff it is full, faithful and "essentially surjective," meaning its image contains an element of every \cong class in \mathcal{D} .

5. All important cat theory notions are preserved

Example 1



\mathcal{C}



\mathcal{D}

$$\mathcal{F}(c) = d_1$$

$\mathcal{F}(1_c) = 1_{d_1}$ is an equivalence.

but $\mathcal{C} \neq \mathcal{D}$

Ex 2 Modules for R and for $M_n(R)$ are equivalent categories

$M \in R\text{-mod} \implies$ col vectors
of length
 n from M

Def Two rings R, S are Morita equivalent if
 $\text{mod-}R$ and $\text{mod-}S$ are (additively) equivalent categories.

Ex 3 \mathbb{C} Ring and affine schemes are equiv categories

$R \longrightarrow$ Prime ideal
Spectrum

Products, Coproducts, universals

Def on HW for products, coproducts.

Then They are! up to \cong if they exist.

Pf Exerc using univ property.

Def $I \in \mathcal{C}$ is universal (or initial) if for each object $C \in \mathcal{C}$ there is a unique morphism $I \rightarrow C$.

$T \in \mathcal{C}$ is couniversal (or terminal) if for each $C \in \mathcal{C}$ there is unique morphism $C \rightarrow T$.

Ex 1. $\{e\}$ initial + terminal in Grp

2. Any two universals are \cong (Proof)

3. Let $\{A_i \mid i \in I\} \subseteq \text{ob } \mathcal{C}$. Define a new category \mathcal{E} with

$\text{ob } \mathcal{E} = \text{pairs } (B, \{f_i \mid i \in I\})$ where
 $B \in \mathcal{C}$ and $f_i: B \rightarrow A_i$

morphism $(B, \{f_i\}) \rightarrow (D, \{g_i\})$ is

a morphism $h: B \rightarrow D$ so $f_i = g_i \circ h$.

Exercise

If $(\prod A_i, \{p_i\})$ exists it is

a terminal object in category \mathcal{E} .

Similarly coproduct is a universal in opp. cat.