p. 668 #5, #20, 22, 24

For forseeable future, R comm ring w/ 1

Review: R is Noetherian if it satisfies the ACC on ideals,
given $I_1 \subseteq I_2 \subseteq \dots \exists k$ s.t. $I_k = I_{k+1} = \dots$ (DCC called Artinian, much stronger! DCC $\to$ ACC)

Thm
1. R Noeth $\Rightarrow$ R/I Noeth. (Corr Thm)
2. TFAE
   a. R Noeth
   b. Every nonempty set of ideals has max elt.
   c. Every ideal f.g.

Hilbert Basis Thm R Noeth $\Rightarrow R[x]$ is

Proof
Suppose $I \subsetneq R[x]$ not f.g. Construct $p_1(x), p_2(x), p_3(x)$ etc.. in I such that p_i has minimal degree among elts in $I - (p_1(x), p_2(x), \dots, p_{i-1}(x))$! Let a_i be leading coef of $p_i(x)$

Def $L = (a_1, a_2, \dots) \subseteq R$, so L is f.g., say $L = (a_1, a_2, \dots, a_N)$!
Thus:

$$a_{N+1} = \sum r_i a_i. \quad \text{Consider:}$$

$$f(x) = \sum_{i=1}^{n} r_i p_i(x) x^{\deg p_{N+1}(x) - \deg p_i(x)}$$

Note that
1. $\deg f(x) = \deg p_{N+1}(x)$
2. $f(x)$ has Lead coef a_{N+1} and $f(x) \in (p_1(x), \dots, p_N(x))$!
3. Thus $p_{N+1}(x) - f(x) \notin (p_1(x), \dots, p_N(x))$ and has smaller deg $\divideontimes$

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Special Case

$R = k$, $k[x_1, x_2, \ldots, x_n]$ is a k-algebra. (VS w/ mult or comm ring w/ scalars)

Prop F.g. k-algebras $\leftrightarrow$ homo images of $k[x_1, \ldots, x_n]$ $\leftrightarrow$ $k[x_1, \ldots, x_n]/(I)$

Goal Relate Comm Alg $\leftrightarrow$ geometry

Def. Let k be field, $\mathbb{A}^n = \{(a_1, \ldots, a_n) \mid a_i \in k\}$ affine n-space.

- like vector space with origin forgotten
- can be done axiomatically

Def. $k[\mathbb{A}^n] = k[x_1, x_2, \ldots, x_n]$ coordinate ring of $\mathbb{A}^n$, functions

Def. Let $S \subseteq k[\mathbb{A}^n]$ Define

$$Z(S) = \{(a_1, a_2, \ldots, a_n) \mid f(a_1, a_2, \ldots, a_n) = 0 \ \forall f \in S\}$$
$$= \text{zero set.}$$

Def. $V \subseteq \mathbb{A}^n$ is an (affine) algebraic variety if $V = Z(S)$ some $S \subseteq k[\mathbb{A}^n]$

Rmks

1. HBT says S is finite WLOG

2. AKA algebraic set, algebraic variety means irreducible algebraic variety - not union of two prop.

Ex $S = \{xy\} \subseteq k[\mathbb{A}^2]$ $V =$ 

$V =$ union of 2, not irreducible.

Examples of algebraic Varieties

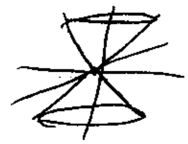
1. $\mathbb{A}^n$ $S=\{0\}$, $\emptyset$, ($S=\{1\}$

2. Varieties in $\mathbb{A}^1 \longleftrightarrow$ finite sets of points or k.

3. Any finite set in $\mathbb{A}^n$ (exercise)

4. If $|S|=1$ then $Z(S)=Z(p(x))$ is called a hypersurface

5. $S=\{z-x^2-y^2\}$



not smooth

Elementary Properties

1. If $(S)=I$ then $Z(S)=Z(I)$, so WLOG take S to be an ideal, HBT says alg variety is $\cap$ of finite # hypersurfaces

2. $Z(S)\cap Z(T)=Z(S\cup T)$, also $\bigcap_i Z(S_i)$

3. $Z(I)\cup Z(J)=Z(IJ)$

4. $I\subseteq J \to Z(J)\subseteq Z(I)$

<u>Warning</u> Z: Ideals in $k[\mathbb{A}^n] \to$ alg varieties is not 1-1.

Ex $Z(x)=Z(x^2)=\{x=0\}$

but $(x^2)\subsetneq(x)$.

Going in Reverse

Let $A \subseteq \mathbb{A}^n$. Def: $\mathcal{I}(A) = \{p(x) \in k[\mathbb{A}^n] \mid p(a_1, \ldots, a_n) = 0 \ \forall (a_1, a_2, \ldots, a_n) \in A\}$

Rmk $\mathcal{I}(A)$ is clearly an ideal, and is the largest ideal of functions which are zero on A.

Ex 1 $I = (x)$, $J = (x^2)$ $\longrightarrow$ $Z(I) = \{x=0\}$, $Z(J) = \{x=0\}$ $\Longrightarrow$ $\mathcal{I}(\{x=0\}) = (x)$

Ex Recall given $(a_1, a_2, \ldots, a_n) = \bar{a}$, $ev_{\bar{a}} : k[x_1, x_2, \ldots, x_n] \to k$.
Thus
$$\mathcal{I}((a_1, a_2, \ldots, a_n)) = \ker ev_A \text{ is maximal ideal}$$
$$= (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$$

Rmk Variety $\mathbb{A}^n \longrightarrow$ coord ring $k[\mathbb{A}^n] \to \{\text{max ideals}\}$ is $\mathbb{A}^n$

Easy Properties Let $A, B \subseteq \mathbb{A}^n$.

1. $A \subseteq B \Rightarrow \mathcal{I}(B) \subseteq \mathcal{I}(A)$
2. $\mathcal{I}(A \cup B) = \mathcal{I}(A) \cap \mathcal{I}(B)$
3. $\mathcal{I}(\emptyset) = k[x_1, \ldots, x_n]$, $\mathcal{I}(\mathbb{A}^n) = 0$ if k is infinite!

   Ex $k = \mathbb{F}_2 = \mathbb{A}^1$ $x^2 + 1 \in \mathcal{I}(\mathbb{A}^1)$

   k ∞, polys can't have only many roots
4. $A \subseteq Z(\mathcal{I}(A))$ ~~Ide~~ $J \subset \mathcal{I}(Z(J))$
5. Suppose $V = Z(I)$ an alg variety. Then $V = Z(\mathcal{I}(V))$.

   Suppose $I = \mathcal{I}(A)$. Then $\mathcal{I}(Z(I)) = I$

   i.e. $Z\mathcal{I}Z = Z$, $\mathcal{I}Z\mathcal{I} = \mathcal{I}$.

Rmk Thus $Z, \mathcal{I}$ bijections varieties $\longleftrightarrow$ ideals of form $\mathcal{I}(V)$ ($k = \bar{k}$ radical ideals)

Def. Let $V \subseteq \mathbb{A}^n$ be alg variety. $k[\mathbb{A}^n]/\mathcal{I}(V) := k[V]$ is the coordinate ring of V.

Rmk. Think of $k[V]$ as polynomials on V, i.e. $\overline{P_1(x)} = \overline{P_2(x)}$ iff $P_1|_V = P_2|_V$

Maps between Alg Varieties

Def Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be varieties

$\varphi: V \to W$ is a regular map if $\exists\ \varphi_1, \varphi_2, \ldots, \varphi_m \in k[x_1, \ldots, x_n]$ s.t.

$$\varphi((a_1, a_2, \ldots, a_n)) = (\varphi_1(a_1, a_2, \ldots, a_n), \ldots, \varphi_m(a_1, a_2, \ldots, a_n)) \quad \forall (a_1, \ldots, a_n) \in V,$$

i.e. φ is a polynomial map. WARNING $\cong$ requires φ 1-1, onto and φ^{-1} is regular

Exercise Let φ be as above. Then

$$\tilde{\varphi}: k[W] \to k[V], \quad \tilde{\varphi}(f) = f \circ \varphi$$

is well defined alg homo.

Thm Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ alg varieties $\exists$ bijection

$$\left\{ \begin{matrix} \text{regular maps} \\ V \to W \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} k\text{-alg homo} \\ k[W] \to k[V] \end{matrix} \right\} \quad \text{that is}$$

1. Contravariant functor
2. "Eq of categories"

Remarks

1. Can think of regular maps $V \to W$ (geometry) purely algebraically

2. Prop Let $\varphi: V \to W$ map of varieties

φ is regular $\leftrightarrow$ $\forall f \in R[W]$ $f \circ \varphi \in R[V]$

In particular $R[V]$ is WD ind. of embedding

3. Groebner Bases, powerful computing tools available