terminal positions. The first selection of a turning arc must be at the largest acute angle in order to minimize the arc, and hence, to minimize the path.

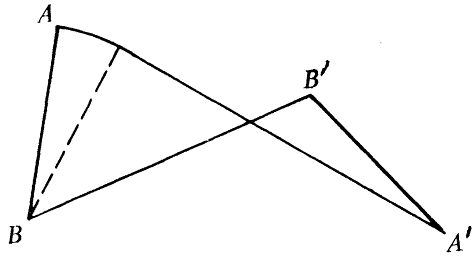

FIG. 7. Crossed joins, one pivot point.

6. The restricted problem with crossed joins and close terminal positions. In each of the foregoing cases, the motion is composed of a combination of rotations and translations. Figure 8, however, shows a case in which AA' crosses BB', and the positions AB and $A'B'$ are so close that tangents cannot be drawn to the turning arcs as in Figures 6 and 7. Then, it is necessary to use pure rotation of the line AB about a center P which is at the intersection of the perpendicular bisectors of AA' and BB'.

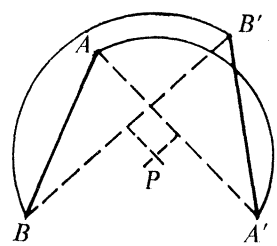

FIG. 8. Crossed joins, pure rotation.

A PRINCIPAL IDEAL RING THAT IS NOT A EUCLIDEAN RING

JACK C. WILSON, University of North Carolina at Asheville.

1. Introduction. In introductory algebra texts it is commonly proved that every Euclidean ring is a principal ideal ring. It is also usually stated that the converse is false, and the student is often referred to a paper by T. Motzkin [**1**]. Unfortunately, this reference does not contain all of the details of the counterexample, and it is not easy to find the remaining details from the references given in Motzkin's paper. The object of this article is to present the counterexample in complete detail and in a form that is accessible to students in an undergraduate algebra class.

Not all authors use precisely the same definitions for these two types of rings. Throughout this paper the following definitions will hold.

DEFINITION 1. *An integral domain R is said to be a Euclidean ring if for every* $x \neq 0$ *in R there is defined a nonnegative integer* $d(x)$ *such that:*

(i) *For all x and y in R, both nonzero,* $d(x) \leqq d(xy)$.

(ii) *For any x and y in R, both nonzero, there exist z and w in R such that* $x = zy + w$ *where either* $w = 0$ *or* $d(w) < d(y)$.

DEFINITION 2. *An integral domain R with unit element is a principal ideal ring if every ideal in R is a principal ideal; i.e., if every ideal A is of the form* $A = (x)$ *for some x in R.*

The ring, R, to be considered is a subset of the complex numbers with the usual operations of addition and multiplication:

$$R = \{a + b(1 + \sqrt{-19})/2 \mid a \text{ and } b \text{ are integers}\}.$$

It is elementary to show that R is an integral domain with unit element. The purpose of this article then is to show that R is a principal ideal ring, but that it is impossible to define a Euclidean norm on R so that with respect to that norm R is a Euclidean ring.

2. The ring is a principal ideal ring. In R there is the usual norm, $N(a + bi) = a^2 + b^2$, which has the property that $N(xy) = N(x)N(y)$ for all complex numbers x and y. In R this norm is always a nonnegative integer. The essential theorem for this part of the example is due to Dedekind and Hasse, and the proof is taken from [**2**, p. 100].

THEOREM 1. *If for all pairs of nonzero elements x and y in R with* $N(x) \geqq N(y)$, *either* $y \mid x$ *or there exist z and w in R with* $0 < N(xz - yw) < N(y)$, *then R is a principal ideal ring.*

Proof. Let $A \neq (0)$ be an ideal in R. Let y be an element of A with minimal nonzero norm, and let x be any other element of A. For all z and w in R, $xz - yw$ is in A so that either $xz - yw = 0$ or $N(xz - yw) \geqq N(y)$. Hence the assumed conditions on R require that $y \mid x$; i.e., $A = (y)$.

The ring R under consideration will now be shown to satisfy the hypotheses of Theorem 1. Observe that $0 < N(xz - yw) < N(y)$ if and only if $0 < N[(x/y)z - w] < 1$. Given x and y in R, both nonzero and $y \nmid x$, write x/y in the form $(a + b\sqrt{-19})/c$ where a, b, c are integers, $(a, b, c) = 1$, and $c > 1$. First of all, assume that $c \geqq 5$. Choose integers d, e, f, q, r such that $ae + bd + cf = 1$, $ad - 19\,be = cq + r$, and $|r| \leqq c/2$. Set $z = d + e\sqrt{-19}$ and $w = q - f\sqrt{-19}$. Thus,

$$\begin{aligned}(x/y)z - w &= (a + b\sqrt{-19})(d + e\sqrt{-19})/c - (q - f\sqrt{-19}) \\ &= r/c + \sqrt{-19}/c.\end{aligned}$$

This complex number is not zero and has norm $(r^2 + 19)/c^2$, which is less than 1 since $|r| \leqq c/2$ and $c \geqq 5$. The only case that is not immediately obvious is $c = 5$, but then $|r| \leqq 2$ so that $r^2 + 19 \leqq 23 < c^2$.

The remaining possibilities are $c = 2$, 3, or 4. Consider these in order:

(i) If $c = 2$, $y \nmid x$ and $(a,b,c) = 1$ imply that a and b are of opposite parity. Set $z = 1$ and $w = [(a-1) + b\sqrt{-19}]/2$ which are elements of R. Thus, $(x/y)z - w = 1/2 \neq 0$ and has norm less than 1.

(ii) If $c = 3$, $(a,b,c) = 1$ implies that $a^2 + 19b^2 \equiv a^2 + b^2 \not\equiv 0 \pmod 3$. Let $z = a - b\sqrt{-19}$ and $w = q$ where $a^2 + 19b^2 = 3q + r$ with $r = 1$ or 2. Thus, $(x/y)z - w = r/3 \neq 0$ and has norm less than 1.

(iii) If $c = 4$, a and b are not both even. If they are of opposite parity, $a^2 + 19b^2 \equiv a^2 - b^2 \not\equiv 0 \pmod 4$. Let $z = a - b\sqrt{-19}$ and $w = q$, where $a^2 + 19b^2 = 4q + r$ with $0 < r < 4$. Thus, $(x/y)z - w = r/4 \neq 0$ and has norm less than 1. If a and b are both odd, $a^2 + 19b^2 \equiv a^2 + 3b^2 \not\equiv 0 \pmod 8$. Let $z = (a - b\sqrt{-19})/2$ and $w = q$, where $a^2 + 19b^2 = 8q + r$ with $0 < r < 8$. Thus, $(x/y)z - w = r/8 \neq 0$ and has norm less than 1.

This completes the proof that R is a principal ideal ring.

3. The ring is not a Euclidean ring. This part of the counterexample is taken from [1]. The material is repeated and slightly elaborated here in order to give a self-contained result accessible to an undergraduate class. As with the previous section the results are stated within the context of the ring R under consideration, but the theorem applies to more general integral domains. Throughout this section R_0 will denote the set of nonzero elements of R.

DEFINITION 3. *A subset P of R_0 with the property $PR_0 \subset P$; i.e., xy is an element of P for all x in P and y in R_0, is called a product ideal of R. (Notice that R_0 is a product ideal.)*

DEFINITION 4. *If S is a subset of R, the derived set of S, denoted by S', is defined by $S' = \{x \in S \mid y + xR \subset S$, for some y in $R\}$.*

LEMMA 1. *If S is a product ideal, then S' is a product ideal.*

Proof. If x is in S', then x is in S and there exists y in R such that $y + xR \subset S$. Let z be in R_0. Since S is a product ideal and x is in S, xz is in S. Further, $y + (xz)R \subset y + xR \subset S$. This shows that $S'R_0 \subset S'$; i.e., S' is a product ideal.

LEMMA 2. *If $S \subset T$, then $S' \subset T'$.*

Proof. If x is in S', then x is in S and hence in T, and there exists a y in R such that $y + xR \subset S \subset T$. Therefore, x is in T', and $S' \subset T'$.

THEOREM 2. *If R is a Euclidean ring, then there exists a sequence, $\{P_n\}$, of product ideals with the following properties:*

(i) $R_0 = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots$,

(ii) $\cap P_n = \varnothing$,

(iii) $P_n' \subset P_{n+1}$, *for each n, and*

(iv) *For each n, $R_0^{(n)}$, the nth derived set of R_0, is a subset of P_n.*

Proof. Let the Euclidean norm in R be symbolized by $d(x)$ for x in R_0. For each nonnegative integer n, define $P_n = \{x \in R_0 \mid d(x) \geqq n\}$. This defines the sequence which obviously has properties (i) and (ii). Suppose that x is in P_n and y is in R_0. $d(xy) \geqq d(x) \geqq n$ which implies that xy is in P_n. This shows that $P_n R_0 \subset P_n$; i.e., for each n, P_n is a product ideal.

For property (iii) let x be in P'_n; i.e., x is in P_n and there exists a y in R such that $y + xR \subset P_n$. Applying the Euclidean algorithm, there exist elements q and r in R with $y = xq + r$ and $r = 0$ or $d(r) < d(x)$. Hence, $r = y + x(-q)$ is in $y + xR \subset P_n$, which implies that $d(r) \geqq n$, and in turn, $d(x) > d(r) \geqq n$, so that $d(x) \geqq n + 1$ and x is in P_{n+1}. This proves property (iii) $P'_n \subset P_{n+1}$.

For property (iv), clearly $R_0 = P_0$ and application of (ii) gives $R'_0 = P'_0 \subset P_1$. Assuming that $R_0^{(n)} \subset P_n$, Lemma 2 and (iii) yield $R_0^{(n+1)} \subset P'_n \subset P_{n+1}$. By induction, (iv) is proved.

COROLLARY. *If $R'_0 = R''_0 \neq \varnothing$, then R is not a Euclidean ring.*

Proof. The hypotheses of the corollary imply that for all n, $R_0^{(n)} = R'_0$. If R is a Euclidean ring, the theorem would require $R'_0 = \cap R_0^{(n)} \subset \cap P_n = \varnothing$.

This corollary is now used to show that R is not a Euclidean ring. First R'_0 is determined. If x is a unit in R, say $xy = 1$, and z is an element of R, $z + x(-yz) = 0$ is not in R_0. This shows that units are not in R'_0. If x is not a unit in R, then using $z = -1$, $z + xy \neq 0$ for all y in R, which shows that if x is not zero and not a unit, x is in R'_0. Altogether, R'_0 is precisely the set of elements of R that are neither units nor zero. Notice that the only units of our example R are 1 and -1. Next, in order to determine the elements of R''_0, it is convenient to use the following terminology:

DEFINITION 5. *An element x of R'_0 is said to be a side divisor of y in R provided there is a z in R that is not in R'_0 such that $x \mid (y + z)$. An element x of R_0 is a universal side divisor provided that it is a side divisor of every element of R.*

If x is in R''_0, then x is in R'_0 and there is a y in R such that $y + xR \subset R'_0$; i.e., x never divides $y + z$ if z is zero or a unit. Thus, x is not a side divisor of y, and therefore, not a universal side divisor. Conversely, if x is not in R''_0, and is in R'_0, then for every y in R there exists a w in R with $y + xw$ not in R'_0; i.e., $y + xw$ is zero or a unit, and therefore, x is a side divisor of y. Since this holds for every y in R, x is a universal side divisor. Together, these two arguments show that R''_0 is the set R'_0 exclusive of the universal side divisors. If it can now be shown that R has no universal side divisors, this will show that $R'_0 = R''_0 \neq \varnothing$, and the corollary will complete the proof that R is not a Euclidean ring.

A side divisor of 2 in R must be a nonunit divisor of 2 or 3. In R, 2 and 3 are irreducible, and therefore, the only side divisors of 2 are 2, -2, 3, and -3. On the other hand, a side divisor of $(1 + \sqrt{-19})/2$ must be a nonunit divisor of $(1 + \sqrt{-19})/2$, $(3 + \sqrt{-19})/2$, or $(-1 + \sqrt{-19})/2$. These elements of R have norms of 5, 7, and 5, respectively, while the norms of 2 and 3 and their associates are 4 and 9, respectively. As a result, no side divisor of 2 is also a side divisor of

$(1+\sqrt{-19})/2$, and there are no universal side divisors in R. All of the details of the counterexample are complete.

References

1. T. Motzkin, The Euclidean algorithm, Bull. Amer. Math. Soc., 55 (1949) 1142–1146.
2. H. Pollard, The Theory of Algebraic Numbers, Carus Monograph 9, MAA, Wiley, New York, 1950.

THE U.S.A. MATHEMATICAL OLYMPIAD

Sponsored by the Mathematical Association of America, the first U.S.A. Mathematical Olympiad was held on May 9, 1972. One hundred students participated. The eight top ranking students were: James Saxe, Albany, N.Y.; Thomas Hemphill, Sepulveda, Calif.; David Vanderbilt, Garden City, N.Y.; Paul Harrington, Central Square, N.Y.; Arthur Rubin, West Lafayette, Ind.; David Anick, New Shrewsbury, N.J.; Steven Raher, Sioux City, Iowa; James Shearer, Livermore, Calif. A detailed report on the Olympiad including the problems and solutions will appear in the March 1973 issue of the American Mathematical Monthly.

The second U.S.A. Mathematical Olympiad will be administered on Tuesday, May 1, 1973. Participation is by invitation only. For further particulars, please contact the Chairman of the U.S.A. Mathematical Olympiad Committee, Dr. Samuel L. Greitzer, Mathematics Department, Room 212, Smith Hall, Rutgers University, Newark, N.J. 07102.

GLOSSARY

KATHARINE O'BRIEN, Portland, Maine

Campus disorder

Skew quadrilateral-quadrangle –
extreme polarization
demonstration.

Generation gap

Two-parameter family –
negative orientation
to communication.

Heart transplant

Removable discontinuity –
binary correlation
operation.

Nylon tires

Synthetic substitution –
translation by rotation
transportation.

Popcorn

Iterated kernels –
onto magnification
transformation.

Suburb

Deleted neighborhood –
little inclination
to integration.