LONGEST INCREASING AND DECREASING SUBSEQUENCES

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This paper deals with finite sequences of integers. Typical of the problems we shall treat is the determination of the number of sequences of length n, consisting of the integers $1, 2, \ldots, m$, which have a longest increasing subsequence of length α. Throughout the first part of the paper we will deal only with sequences in which no numbers are repeated. In the second part we will extend the results to include the possibility of repetition. Our results will be stated in terms of standard Young tableaux.

PART I

Definition. A standard Young tableau of order n is an arrangement of n distinct natural numbers in rows and columns so that the numbers in each row and in each column form increasing sequences, and so that there is an element of each row (column) in the first column (row) and there are no gaps between numbers.

Example.

$$\begin{matrix} 2 & 4 & 7 \\ 3 & 8 & \\ 5 & 9 & \end{matrix} \qquad \text{(order} = 7)$$

Definition. The *shape* of a standard tableau is an arrangement of squares with one square replacing each number in the standard tableau.

Example. The shape of $\begin{matrix} 2 & 4 & 7 \\ 3 & 8 & \\ 5 & 9 & \end{matrix}$ is as shown in Figure 1.

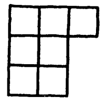

FIG. 1.

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One reason that standard tableaux are so useful to us is that it is easy to compute the number of standard tableaux of a given shape either by means of a simple recurrence relation, or by means of the following elegant result; Frame, Robinson, and Thrall **(1)**.

THEOREM. *The number of standard tableaux of a given shape containing the integers* 1, 2, . . . , n *is*

$$\frac{n!}{\prod_{j=1}^{n} h_j} \tag{1}$$

Here the h_j are the hook lengths, that is, the number of elements counting from the bottom of a column to a given element and then to the right end of the row.

Example. To compute the number of standard tableaux of the shape shown in Figure 2(a), we first find the hook lengths, which are shown in Figure

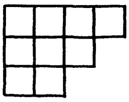

FIG. 2(a).

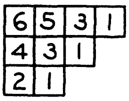

FIG. 2(b).

2(b). Then we find that the number of standard tableaux of this shape is

$$\frac{9!}{6\cdot 5\cdot 3\cdot 1\cdot 4\cdot 3\cdot 1\cdot 2\cdot 1} = 168.$$

Definition. $S \leftarrow x$ is defined as the array obtained from the standard tableau, S, by means of the following steps:

(i) Insert x in the first row of S either by displacing the smallest number which is larger than x, or if no number is larger than x, by adding x at the end of the first row.

(ii) If x displaced a number from the first row, then insert this number in the second row either by displacing the smallest number which is larger than it or by adding it at the end of the second row.

(iii) Repeat this process row by row until some number is added at the end of a row.

In the above steps "adding at the end of the row" is interpreted as putting in the first column in the given row if the row does not yet have any entries in it. We define $x \rightarrow S$ similarly except that we replace the word "row" by the word "column" throughout.

Example. If

$$S = \begin{matrix} 2\,4\,7 \\ 3\,8 \\ 5\,9 \end{matrix}$$

then

$$S \leftarrow 6 = \begin{matrix} 2\,4\,6 \\ 3\,7 \\ 5\,8 \\ 9 \end{matrix} \quad \text{and} \quad 6 \rightarrow S = \begin{matrix} 2\,4\,7 \\ 3\,8 \\ 5\,9 \\ 6 \end{matrix}$$

LEMMA 1. *$S \leftarrow x$ and $x \rightarrow S$ are standard tableaux.*

Proof. Since the proofs for $S \leftarrow x$ and $x \rightarrow S$ are similar we consider only $S \leftarrow x$.

First we note that if two consecutive rows of S have the same length, and if a number is displaced from the first of these two rows, then it will either displace the number which was standing under it or else some number to its left, and thus will not be added at the end of the row. Thus a row cannot be made longer than the row above it and $S \leftarrow x$ cannot fail to be a standard tableau on account of its shape. Thus we have only to prove that the numbers in each row and column still form increasing sequences.

A number is inserted into a row in such a place that the number to its left (if any) is smaller, and the number to its right (if any) is larger. Thus the numbers in each row form increasing sequences.

The number (if any) which ends up below a number which is inserted at a new position is either the number which it displaced, which is therefore larger, or else the number which previously stood below the number which it displaced, which is larger still.

When a number is displaced from one row to the next it ends up either in the position directly beneath the one in which it originally stood, or else further to the left (since it is smaller than the number which previously stood underneath it). Thus it is either under the number which displaced it, which is therefore smaller, or else a number to the left of it, which is smaller still.

The last two paragraphs show that two consecutive numbers in a column form an increasing sequence if either of them has just been inserted into its present position. If neither of them has just been inserted, then they are the numbers which were previously there in S and which therefore are in increasing order. Hence the columns also form increasing sequences and the proof of the lemma is completed.

Definition. The *P-symbol* corresponding to a sequence of distinct integers $x_1x_2 \ldots x_n$ is the standard tableau $(\ldots((x_1 \leftarrow x_2) \leftarrow x_3) \ldots \leftarrow x_n)$. The *Q-symbol* corresponding to the same sequence is the array which is obtained by putting k in the square which is added to the shape of the P-symbol when x_k is inserted in the P-symbol.

Examples.

Sequence	3	3 5	3 5 4	3 5 4 9	3 5 4 9 8	3 5 4 9 8 2	3 5 4 9 8 2 7
P-symbol	3	3 5	3 4	3 4 9	3 4 8	2 4 8	2 4 7
			5	5	5 9	3 9	3 8
						5	5 9
Q-symbol	1	1 2	1 2	1 2 4	1 2 4	1 2 4	1 2 4
			3	3	3 5	3 5	3 5
						6	6 7

LEMMA 2. *The Q-symbol corresponding to an arbitrary sequence is a standard tableau.*

Proof. Since the Q-symbol has the same shape as the P-symbol, and since the P-symbol is a standard tableau, the shape of the Q-symbol is legitimate. Each digit added to the Q-symbol is larger than all of the previous digits, and in particular is larger than the digits above it and to its left. Hence the numbers in each row and column form increasing sequences, and the lemma is established.

LEMMA 3. *There is a one-to-one correspondence between sequences made with the n distinct integers* $x_1, x_2, \ldots, x_n$ *and ordered pairs of standard tableaux of the same shape—the first containing* $x_1, x_2, \ldots, x_n$ *and the second containing* $1, 2, \ldots, n$.

Proof. Given a sequence, the P-symbol and Q-symbol are uniquely determined standard tableaux of the type mentioned in the lemma. Given a pair of standard tableaux of the appropriate types we can find the unique sequence which could have them for a P-symbol and Q-symbol as follows: The position of the largest number in the second tells us which number was added on to a row of the first without displacing another number when the last digit was inserted. This must have been displaced from the previous row by the largest number which is smaller than it (there always will be at least one number smaller than it in the preceding row since the one directly above it is smaller). This in turn must have been displaced from the next row up. Finally we get to the first row and discover what number was inserted into it. This is the last digit of the sequence. We now also know what the P-symbol and Q-symbol were before the last digit was inserted. Thus we can repeat the procedure to find the next to the last digit of the sequence. This proves the lemma.

Note. Since there are $n!$ possible sequences of $x_1, x_2, \ldots, x_n$, Lemma 3 shows that there are $n!$ ordered pairs of standard tableaux of order n such that the shapes of tableaux in each pair are the same, but the shapes of tableaux in different pairs are not necessarily the same. This fact is already known **(2)**. Of course, the number of ordered pairs of standard tableaux of a given shape is equal to the square of the number of standard tableaux of that shape, which is given in turn by Expression (1).

Definition. The jth *basic subsequence* of a given sequence consists of the digits which are inserted into the jth place in the first row of the P-symbol.

LEMMA 4. *Each basic subsequence is a decreasing subsequence.*

Proof. Each number in the jth basic subsequence, on insertion in the first row displaces the previous member of the jth basic subsequence, which must therefore be larger than the present member.

LEMMA 5. *Given any member of the* jth *basic subsequence, we can find a member of the* $(j - 1)$st *basic subsequence which is smaller and which occurs further to the left in the given sequence.*

Proof. The number in the $(j - 1)$st place in the first row, when the given member of the jth basic subsequence is inserted, *is* such a member of the $(j - 1)$st basic subsequence.

THEOREM 1. *The number of columns in the P-symbol (or the Q-symbol) is equal to the length of the longest increasing subsequence of the corresponding sequence.*

Proof. The number of columns is the same as the number of basic subsequences. By Lemma 4 there can be at most one member of each basic subsequence in any increasing subsequence. By Lemma 5 we can construct an increasing subsequence with one element from each basic subsequence, Q.E.D.

Note. The proof shows us how to actually obtain in increasing subsequence of maximal length.

LEMMA 6. $(x \rightarrow S) \leftarrow y = x \rightarrow (S \leftarrow y)$.

Proof. Suppose first, that of all the digits in x, y, and S, the largest is y. We represent S schematically by Figure 3. There are two cases of interest.

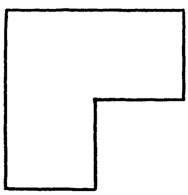

FIG. 3.

The square added to the shape of S in $x \rightarrow S$ is in the first row, or it is not. We represent $x \rightarrow S$ schematically in these two cases by Figure 4(a) and 4(b) respectively, where x' is the number added to the end of some column without displacing another number when we form $x \rightarrow S$. It is easily verified

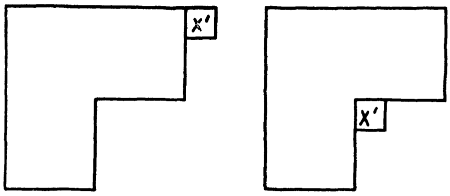

FIG. 4(a). FIG. 4(b).

that in the first case the final result is as shown in Figure 5(a) and in the second case the result is that of Figure 5(b).

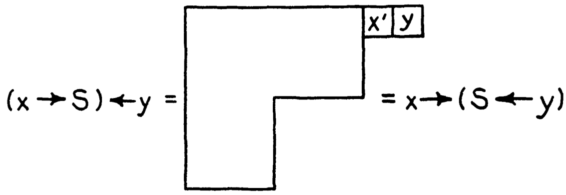

FIG. 5(a).

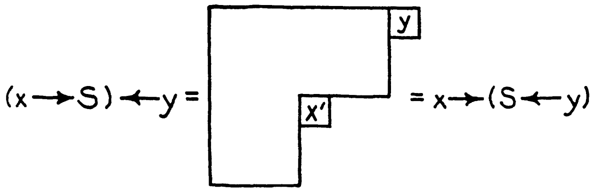

FIG. 5(b).

This proves the lemma if y is the largest number involved, and the proof is similar if x is the largest number involved.

Suppose now that, of all the digits in x, y, and S, the largest is N, and that N is in S. In this case we use induction. The lemma can be easily verified by direct calculation if S is of order 0, 1, or 2. We assume the lemma true for S of order n, and prove that it is then true for S of order $n + 1$.

Let us suppose, then, that S is of order $n + 1$. Now, since N is the largest number in S, we see that N is at the end of whatever row it is in, and also at the end of its column. Thus, if we remove N from S we will obtain a new standard tableau, S', of order n. Now since N is larger than any of the other numbers, it can never displace any of them, and hence the presence or absence of N cannot have any influence on the position of the other numbers. Thus $(x \to S) \leftarrow y$ will be the same as $(x \to S') \leftarrow y$ except that N is added somewhere, and $x \to (S \leftarrow y)$ will be the same as $x \to (S' \leftarrow y)$ except for the addition of N. However, since S' is of order n, we have by assumption

$$(x \to S') \leftarrow y = x \to (S' \leftarrow y).$$

Thus we have only to prove that N occupies the same position in $(x \rightarrow S) \leftarrow y$ and $x \rightarrow (S \leftarrow y)$ to prove the lemma. The truth of this can be easily verified for each of the possible cases which can arise as to the relative locations of N, x', and y'. Here $x'(y')$ is the number which is added to some column (row) without displacing another number when we form $x \rightarrow S'(S' \leftarrow y)$. In making these verifications it is necessary to keep the following facts in mind.

If x' and y' do not fall into the same square, then we represent S', $x \rightarrow S'$, and $S' \leftarrow y$ schematically by Figure 6(a), 6(b), and 6(c) respectively. The shape of $(x \rightarrow S') \leftarrow y$ must have a square added to the shape of

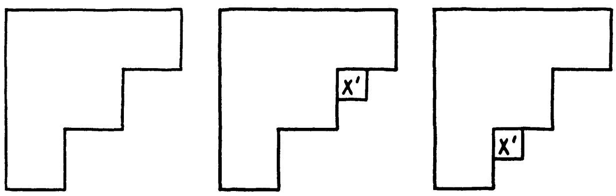

FIG. 6(a). FIG. 6(b). FIG. 6(c).

$x \rightarrow S'$, and the shape of $x \rightarrow (S' \leftarrow y)$ must have a square added to the shape of $S' \leftarrow y$. By assumption $(x \rightarrow S') \leftarrow y = x \rightarrow (S' \leftarrow y)$ so that the shape of $(x \rightarrow S') \leftarrow y$ and $x \rightarrow (S' \leftarrow y)$ must be Figure 7.

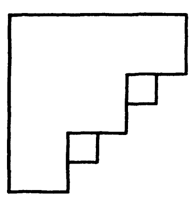

FIG. 7.

If x' (in $x \rightarrow S'$) and y' (in $S' \leftarrow y$) occupy the same position then we schematically represent S', $x \rightarrow S'$, and $S' \leftarrow y$ by Figure 8(a), 8(b), and 8(c) respectively. Here the shaded parts of $x \rightarrow S'$ and $S' \leftarrow y$ are the

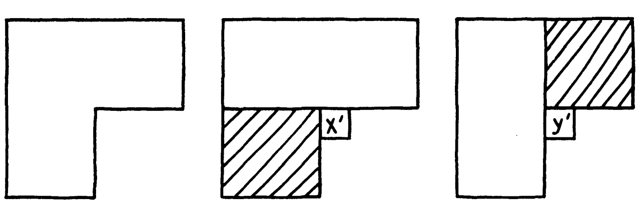

FIG. 8(a). FIG. 8(b). FIG. 8(c).

regions where numbers could have been displaced. Now let us suppose that $y' > x'$. Then when we insert y into $x \rightarrow S'$ the same numbers will be displaced in each row as were displaced when we inserted y into S, until we displace y'.

In $S' \leftarrow y$ we would have put y' where x' is, but $y' > x'$, thus y' will be added at the end of the row containing x', and the shape of $(x \rightarrow S') \leftarrow y$ (and hence of $x \rightarrow (S' \leftarrow y)$) will be Figure 9. If we had had $x' > y'$, then

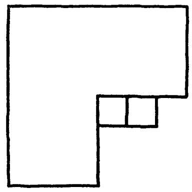

FIG. 9.

the shape of $(x \rightarrow S') \leftarrow y$ and $x \rightarrow (S' \leftarrow y)$ would have been Figure 10. Thus, if we know the shapes of $x \rightarrow S'$ and $S' \leftarrow y$, and if we know whether $x' > y'$ or $x' < y'$, then we know the shape of $(x \rightarrow S') \leftarrow y$ and $x \rightarrow (S' \leftarrow y)$.

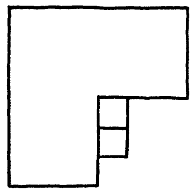

FIG. 10.

Now we can return to the problem of showing that N has the same position in $(x \rightarrow S) \leftarrow y$ and $x \rightarrow (S \leftarrow y)$. As we mentioned there are several special cases. We will consider only three of these as the others go in the same way. First suppose that the position of N in S does not coincide with either the position of x' in $x \rightarrow S'$ or the position of y' in $S' \leftarrow y$. Then N will never be displaced and it will have the same position in $(x \rightarrow S) \leftarrow y$ and $x \rightarrow (S \leftarrow y)$ as it does in S.

Next suppose that the position of N in S coincides with the position of x' in $x \rightarrow S'$, and that the position of y' in $S' \leftarrow y$ lies to the left of this. Then we have schematically Figure 11.

Finally suppose that the position of N in S coincides with the position of x' in $x \rightarrow S'$, and that the position of y' in $S' \leftarrow y$ lies one column to the right of this. Then schematically we have Figure 12. Proceeding similarly we can verify all of the other special cases, and hence the validity of Lemma 6.

LEMMA 7. *If one sequence is a second sequence written backwards, then P-symbol of the first is obtained from the P-symbol of the second by interchanging rows and columns.*

Proof. First we note that $x \rightarrow y = x \leftarrow y$ since if $x < y$ they are both xy and if $x > y$ they are both $\begin{smallmatrix} y \\ x \end{smallmatrix}$. Now we define $P(x_1, x_2, \ldots, x_n) \equiv (\ldots((x_1$

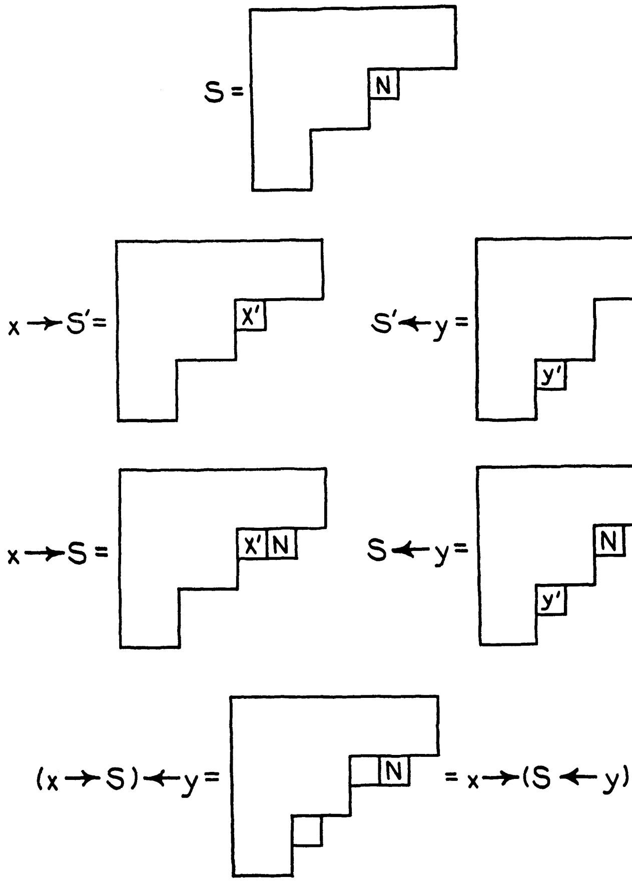

FIG. 11.

$\leftarrow x_2) \leftarrow x_3) \ldots \leftarrow x_n)$ and $\tilde{P}(x_1, x_2, \ldots, x_n) \equiv (x_1 \rightarrow \ldots (x_{n-2} \rightarrow (x_{n-1} \rightarrow x_n)) \ldots)$. Next we assume that $P(x_1, x_2, \ldots, x_{n-1}) = \tilde{P}(x_1, x_2, \ldots, x_{n-1})$ and that $P(x_1, x_2, \ldots, x_n) = \tilde{P}(x_1, x_2, \ldots, x_n)$ and prove that $P(x_1, x_2, \ldots, x_n, x_{n+1}) = \tilde{P}(x_1, x_2, \ldots, x_n, x_{n+1})$. (We have just shown that $P(x_1, x_2) = x_1 \leftarrow x_2 = x_1 \rightarrow x_2 = \tilde{P}(x_1, x_2)$, furthermore $P(x_1) = x_1 = \tilde{P}(x_1)$.) We have

$$\begin{aligned} P(x_1, x_2, \ldots, x_n, x_{n+1}) &= P(x_1, x_2, \ldots, x_n) \leftarrow x_{n+1} \\ &= \tilde{P}(x_1, x_2, \ldots, x_n) \leftarrow x_{n+1} \\ &= [x_1 \rightarrow \tilde{P}(x_2, \ldots, x_n)] \leftarrow x_{n+1} \\ &= x_1 \rightarrow [\tilde{P}(x_2, \ldots, x_n) \leftarrow x_{n+1}] \\ &= x_1 \rightarrow [P(x_2, \ldots, x_n) \leftarrow x_{n+1}] \\ &= x_1 \rightarrow P(x_2, \ldots, x_n, x_{n+1}) \\ &= x_1 \rightarrow \tilde{P}(x_2, \ldots, x_n, x_{n+1}) \\ &= \tilde{P}(x_1, x_2, \ldots, x_n, x_{n+1}). \end{aligned}$$

Of these lines, the second, fifth, and seventh follow by assumption, and the

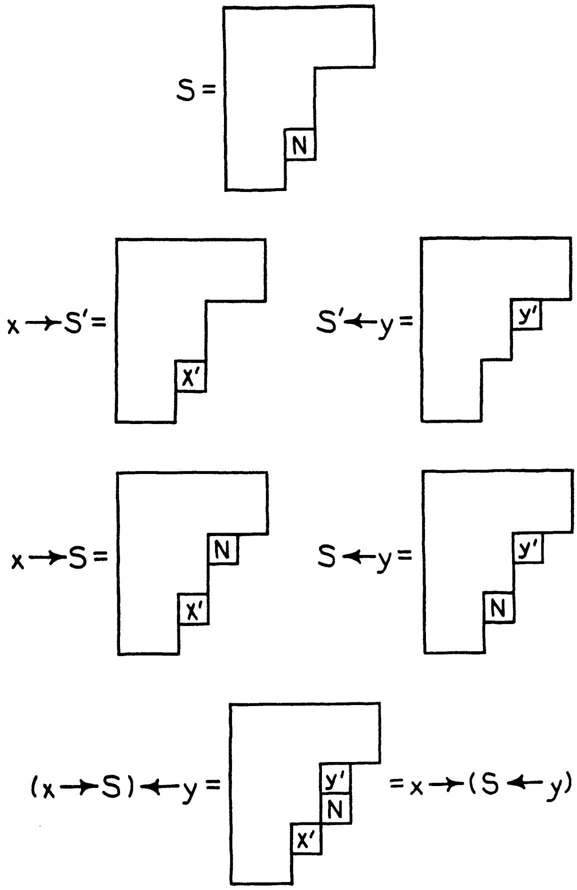

FIG. 12.

fourth from Lemma 6. Now $P(x_1, \ldots, x_n)$ is the P-symbol for the sequence $x_1, x_2, \ldots, x_n$, while $\tilde{P}(x_1, x_2, \ldots, x_n)$ is the P-symbol for the sequence $x_n, \ldots, x_2, x_1$ with rows and columns interchanged. Hence the lemma follows.

Note. It must not be assumed that Lemma 7 holds for Q-symbols.

THEOREM 2. *The number of rows in the P-symbol (or the Q-symbol) is equal to the length of the longest decreasing subsequence of the corresponding sequence.*

Proof. This follows immediately from Theorem 1 and Lemma 7, since writing a sequence backwards changes increasing subsequences into decreasing subsequences.

THEOREM 3. *The number of sequences consisting of the distinct numbers $x_1, x_2, \ldots, x_n$, and having a longest increasing subsequence of length α and a longest decreasing subsequence of length β, is the sum of the squares of the numbers of standard tableaux with shapes having α columns and β rows.*

Proof. Follows immediately from Lemma 3 and Theorems 1 and 2 (see also the note to Lemma 3).

Example. To find the number of permutations of $1, 2, 3, \ldots, 25$ having a longest decreasing subsequence of length three and a longest increasing subsequence of length 21 we note that the only allowed shapes with 25 squares, 21 columns, and 3 rows are those of Figure 13.

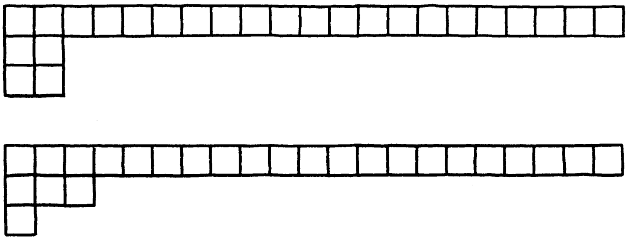

FIG. 13.

By the Frame–Robinson–Thrall theorem, the corresponding numbers of standard tableaux are 21,000 and 31,350 respectively. Thus the desired number of permutations is

$$21{,}000^2 + 31{,}350^2 = 1{,}423{,}822{,}500.$$

PART II

We now want to consider sequences in which some of the numbers are repeated. We can obtain the properties of such sequences in terms of sequences without repetitions by a simple artifice. Suppose the smallest number appears p times in the sequence, the next smallest q times, etc. We replace the p occurrences of the smallest number by the numbers $1, 2, \ldots, p$ (in this order), the q occurrences of the next number by $p + 1, p + 2, \ldots, p + q$, etc. Then the decreasing subsequences of the two sequences will be in one-to-one correspondence, while the increasing subsequences of the new sequence will be in one-to-one correspondence with the non-decreasing subsequences of the original sequence.

Example. Given the sequence 3 3 2 3 4 1, we replace 1 by 1, 2 by 2, the three 3's by 4, 5, 6, and 4 by 7. The result is 4 5 2 6 7 1. The latter sequence has a decreasing subsequence 5 2 1 which corresponds to a decreasing subsequence 3 2 1 in the original and an increasing subsequence 4 5 6 7 which corresponds to a non-decreasing subsequence 3 3 3 4 in the original.

If we construct the P-symbol for the derived sequence, and map the numbers in it back to the numbers in the original sequence, then we get a modified standard tableau in which repeated numbers are allowed, the numbers in each column form an increasing sequence, and the numbers in each row form a non-decreasing sequence. Since the numbers in the Q-symbol refer to the order of addition of spaces to the P-symbol, the Q-symbols of the two sequences will be identical.

We can get modified forms of each of the results in Part I. The main result, Theorem 3, now takes the form:

THEOREM 4. *The number of sequences of* $x_1, x_2, \ldots, x_n$ *having a longest non-decreasing sequence of length* α *and a longest decreasing sequence of length* β *is the sum of the products of the number of modified standard tableaux of a given shape with the number of standard tableaux of the same shape, the shapes each having* α *columns and* β *rows.*

Example. To find the number of sequences of seven numbers consisting entirely of 1's, 2's, and 3's having a longest non-decreasing sequence of length four and a longest decreasing sequence of length three, we proceed as follows. The possible tableaux must have the shape of Figure 14.

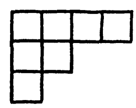

FIG. 14.

The possible modified standard tableaux are

			1 1 1 1 2 2 3	1 1 1 1 2 3 3
1 1 1 2 2 2 3	1 1 1 2 2 3 3	1 1 1 3 2 2 3	1 1 1 3 2 3 3	1 1 2 2 2 2 3
1 1 2 2 2 3 3	1 1 2 3 2 2 3	1 1 2 3 2 3 3	1 1 3 3 2 2 3	1 1 3 3 2 3 3
1 2 2 2 2 3 3	1 2 2 3 2 3 3	1 2 3 3 2 3 3	They are 15 in number.	

By the Frame–Robinson–Thrall theorem the number of standard tableaux of this shape is 35. Hence the number of sequences of the desired type is $15 \times 35 = 525$.

As a further example we will work out explicit formulae for binary sequences (sequences consisting of 0's and 1's). In this case the modified standard tableaux have the general form of Figure 15, where the bracketed region can have any division of 0's and 1's (the 0's preceding the 1's, of course).

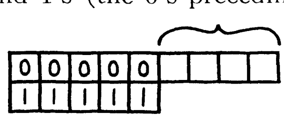

FIG. 15.

Let n be the number of digits in the sequence. Let m be the length of the longest non-decreasing subsequence. Then there are no sequences for which $m < n/2$. If $m = n$ the longest decreasing subsequence is of length 1. If $n/2 \leqslant m < n$, the longest decreasing subsequence is of length 2.

The number of possible modified tableaux is $2m - n + 1$. The number of standard tableaux is

$$(2m - n + 1)\frac{n!}{(m+1)!(n-m)!}.$$

Thus the number of binary sequences of n digits with a longest non-decreasing subsequence of length m is

$$\frac{n!(2m-n+1)^2}{(m+1)!(n-m)!}.$$

Note. Since the total number of binary sequences is 2^n we have

$$2^n = \sum_{m \geqslant n/2}^{n} \frac{n!(2m-n+1)^2}{(m+1)!(n-m)!}.$$

In the above derivation we allowed all possible binary sequences. Theorem 4 also readily solves the problem if the number of 0's and 1's in the sequence is fixed. In this case there is at most one modified tableau and thus the number of sequences of n digits with a longest non-decreasing subsequence of length m is

$$\frac{n!(2m-n+1)}{(m+1)!(n-m)!}$$

with the additional restriction that the number, p, of 0's in the sequence must satisfy $n - m \leqslant p \leqslant m$.

Note. This shows that

$$\binom{n}{p} = \sum_{m=\max(p,n-p)}^{n} \frac{n!(2m-n+1)}{(m+1)!(n-m)!}.$$

Throughout Part II we could have dealt equally well with increasing and non-increasing subsequences rather than decreasing and non-decreasing subsequences.

References

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