Math 464/564 Fall 2017 Homework Number 10- Due Tuesday 11/21/17

Let $A = (a_{ij})_{i,j\geq 1}$ be an integer matrix with finitely many nonzero entries. Suppose A has row and column sums:

$$r_i = \sum_j a_{ij}, c_j = \sum_i a_{ij}.$$

Define the row sum vector by $\mathrm{row}(A) = (r_1, r_2, \ldots)$ and the column sum vector by $\mathrm{col}(A) = (c_1, c_2, \ldots)$. Say A is a $(0,1)$ matrix if all entries are 0 or 1.

1. Consider the expansion of the e_λ in terms of the basis of monomial symmetric functions:

$$e_\lambda = \sum_{\mu\vdash n} M_{\lambda\mu} m_\mu.$$

Prove that $M_{\lambda\mu}$ is the number of $(0,1)$ matrices $A = (a_{ij})$ satisfying $\mathrm{row}\, A = \lambda$ and $\mathrm{col}(A) = \mu$. In particular then $M_{\lambda\mu}$ is zero unless λ and μ partition the same integer.

Hint: These are symmetric functions so $M_{\lambda\mu}$ is just the coefficient of x^μ in e_λ.

2. Let $m_\lambda(x)$ and $m_\mu(y)$ denote monomial symmetric functions in sets of variables $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$. Prove:

$$\prod_{i,j}(1 + x_i y_j) = \sum_{\lambda,\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y)$$

where λ and μ range over all partitions. It suffices to take $|\lambda| = |\mu|$ as otherwise $M_{\lambda\mu}$ is zero by the previous problem.

3. Repeat Exercise 1 except for $N_{\lambda\mu}$ where:

$$h_\lambda = \sum_{\mu\vdash n} N_{\lambda\mu} m_\mu.$$

That is, express $N_{\lambda\mu}$ in terms of matrices with a given property.

4. As is in Problem 2 show that:

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda,\mu} N_{\lambda\mu} m_\lambda(x) m_\mu(y).$$

5. Expand the power series $\prod_{i\geq 1}(1 + x_i + x_i^2)$ in terms of elementary symmetric functions.

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