Name: SOLUTIONS

Math 461/561 Final Exam - December 16, 2010

1. **(20 points)** Complete the following:

a. Let L be a complex Lie algebra. The *Killing form* on L is the symmetric bilinear form defined by...

$$K(x,y) = \text{trace}(\text{ad}\,x \circ \text{ad}\,y)$$

b. A Lie subalgebra H of a Lie algebra L is said to be a *Cartan subalgebra* if ...

1. Every $h \in H$ is semisimple.
2. H is abelian.
3. H is maximal subject to #1 & 2 above.

c. A subset R of a real inner product space E is a *root system* if it satisfies the following axioms:

1. R is finite, spans E and $0 \notin R$
2. If $\alpha \in R$ then $k\alpha \in R \Leftrightarrow k = \pm 1$
3. $\alpha, \beta \in R \Rightarrow \langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$
4. $\alpha, \beta \in R \Rightarrow s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in R$

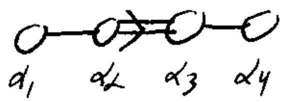
d. Let R be a root system. The *Cartan matrix* of R is defined by...

Choose a base $B = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$

Then $C_{ij} = \langle \alpha_i, \alpha_j \rangle$

2. **(20 points)** True or false. If false, give a counterexample or explanation.

F a. The roots in the F_4 root system all have the same length.



so α_2 and α_3 have different lengths

F b. Let $L = b(3, \mathbb{C})$, the 3×3 upper triangular matrices. Then any finite-dimensional L-module is completely reducible.

The natural module $\mathbb{C}^3$ of column vectors is not completely reducible.

T c. Up to isomorphism, $sl(2, \mathbb{C})$ has a unique irreducible module in each dimension.

F d. A root system is determined up to isomorphism by its Weyl group.

Types B_ℓ and C_ℓ are $\not\cong$ root systems but have the same Weyl group.

T e. Let L be a complex semisimple Lie algebras with CSA H and root system Φ. If Φ is irreducible then L is simple.

F f. The Lie algebra $sl(n, F)$ is simple for any field.

If $\operatorname{char} F = p \mid n$ then $Id_n \in sl(n, F)$ and spans a 1-dimensional ideal

3. **(20 points)** Let $L = gl_S(2l, \mathbb{C})$ where S is the matrix $S = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. We write elements of L as block matrices of shapes adapted to the blocks of S. Calculation shows that:

$$L = \{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : p = p^t, q = q^t. \}$$

Let H be the set of diagonal matrices in L. **Assume** $l = 3$ so your matrices are 6×6.

(1) Show that H is a Cartan subalgebra.

(2) Work out the root space decomposition.

(3) Find a base for the root system and compute the Cartan matrix.

(4) Give the Dynkin diagram.

Let $h = \operatorname{diag}(h_1, h_2, h_3, -h_1, -h_2, -h_3)$ be an arbitrary element of H.

Let

$$X = \begin{pmatrix} m_{11} & m_{12} & m_{13} & p_{11} & p_{12} & p_{13} \\ m_{21} & m_{22} & m_{23} & p_{12} & p_{22} & p_{23} \\ m_{31} & m_{32} & m_{33} & p_{13} & p_{23} & p_{33} \\ q_{11} & q_{12} & q_{13} & -m_{11} & -m_{21} & -m_{31} \\ q_{12} & q_{22} & q_{23} & -m_{12} & -m_{22} & -m_{32} \\ q_{13} & q_{23} & q_{3} & -m_{13} & -m_{23} & -m_{33} \end{pmatrix}$$

be an arbitrary element of L.

Then

$\cancel{0 \quad (h_1-h_2)m_{12} \quad (h_1-h_3)m_{13} \quad 2h_1p_{11} \quad (h_1+h_2)p_{12}}$

$$[h, X] = \left(\begin{array}{ccc|ccc} 0 & (h_1-h_2)m_{12} & (h_1-h_3)m_{13} & 2h_1 p_{11} & (h_1+h_2)p_{12} & (h_1+h_3)p_{13} \\ (h_2-h_1)m_{21} & 0 & (h_2-h_3)m_{23} & (h_1+h_2)p_{12} & 2h_2 p_{22} & (h_2+h_3)p_{23} \\ (h_3-h_1)m_{31} & (h_3-h_2)m_{32} & 0 & (h_1+h_3)p_{13} & (h_2+h_3)p_{23} & 2h_3 p_{33} \\ \hline -2h_1 q_{11} & -(h_1+h_2)q_{12} & -(h_1+h_3)q_{13} & 0 & -(h_2-h_1)m_{21} & -(h_3-h_1)m_{31} \\ -(h_1+h_2)q_{12} & -2h_2 q_{22} & -(h_2+h_3)q_{23} & -(h_1-h_2)m_{12} & 0 & -(h_3-h_2)m_{32} \\ -(h_1+h_3)q_{13} & -(h_2+h_3)q_{3} & -2h_3 q_{33} & -(h_1-h_3)m_{13} & -(h_2-h_3)m_{23} & 0 \end{array}\right)$$

#3 cont.

1. Clearly H is abelian and every element is diagonalizable, so semisimple. Setting $[h, X] = 0$ we obtain all the entries in X are 0 except m_{11}, m_{22}, m_{33}, so $X \in H$. Thus $C_L(H) = H$ so H is a CSA.

2. The "obvious" basis of L is already a root space decomp:

$$L = H \oplus \underline{\text{root spaces}} \quad \text{as below.}$$

root	$\varepsilon_i - \varepsilon_j$ $i \neq j$	$\varepsilon_i + \varepsilon_j$ $i \neq j$	$-(\varepsilon_i + \varepsilon_j)$ $i \neq j$	$2\varepsilon_i$	$-2\varepsilon_i$
root space	m_{ij}	p_{ij}	q_{ij}	p_{ii}	q_{ii}

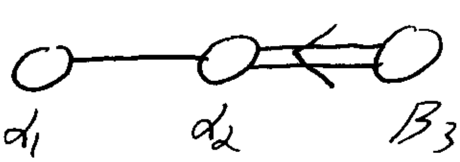
3. Base is $\alpha_1 = \varepsilon_1 - \varepsilon_2$ $\quad \alpha_2 = \varepsilon_2 - \varepsilon_3$ $\quad B_3 = 2\varepsilon_3$

$$\langle \alpha_1, \alpha_2 \rangle = \frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{-2}{2} = -1 \qquad \langle \alpha_1, B_3 \rangle = \frac{2(0)}{4} = 0$$

$$\langle \alpha_2, \alpha_1 \rangle = -1 \qquad \langle \alpha_2, B_3 \rangle = \frac{2(\alpha_2, B_3)}{(B_3, B_3)} = \frac{2 \cdot -2}{4} = -1$$

$$\langle B_3, \alpha_1 \rangle = 0 \qquad \langle B_3, \alpha_2 \rangle = \frac{2(B_3, \alpha_2)}{(\alpha_2, \alpha_2)} = \frac{2 \cdot -2}{2} = -2$$

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

4. 

4. **(20 points)** State Cartan's first Criterion for a complex Lie algebra to be solvable. *Use this criterion* to show that any 2-dimensional Lie algebra is solvable.

Thm L is solvable $\iff \kappa(x,y)=0 \ \forall x\in L, y\in L'$

We proved all 2-dim Lie algs are abelian or $= \langle x,y \mid [x,y]=x\rangle$

L abelian Then $\operatorname{ad} x = 0 \ \forall x \Rightarrow \kappa(x,y)\equiv 0$

Otherwise $\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\operatorname{ad} y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ $\operatorname{ad}x\circ\operatorname{ad}y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\operatorname{ad}y\circ\operatorname{ad}x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Since $L' = \langle x\rangle$ and $\kappa(x,x)=\kappa(y,x)=0$,

L is solvable.

5. **(20 points)** Let $L = sl(3,\mathbb{C})$ and let $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$ be the usual base for the root system. Note: This is 8.3 from HW

(1) Determine the subalgebra $sl(\alpha_1) = \operatorname{span}\{e_{\alpha_1}, f_{\alpha_1}, h_{\alpha_1}\}$ (which is isomorphic to $sl(2,\mathbb{C})$).

(2) Consider L as an $sl(\alpha_1)$ module under the adjoint action. Decompose L as a direct sum of irreducible $sl(\alpha_1)$ modules and identify each summand according to the classification of irreducible $sl(2,\mathbb{C})$ modules.

1. $e_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $f_{\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. Let $V_2 = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = sl(\alpha_1)$

$V_1 = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

$\tilde{V}_1 = \operatorname{span}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$

$V_0 = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right\}$

Clearly $L \cong V_2 \oplus V_1 \oplus \tilde{V}_1 \oplus V_0$ as vector spaces.

We claim each V_i or $\tilde{V}_i$ is an irreducible $\mathfrak{sl}(2,1)$ submodule

$\underline{V_2}$: $\left[\begin{pmatrix} X & 0 \\ & 0 \\ 0\,0 & 0 \end{pmatrix}, \begin{pmatrix} Y & 0 \\ & 0 \\ 0\,0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [X,Y] & 0 \\ & 0 \\ 0\,0 & 0 \end{pmatrix}$ so V_2 is clearly

$\cong$ adjoint module for $\mathfrak{sl}(2,1)$, the 3-dim irreducible

$\underline{V_1}$ $\left[\begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & ax+by \\ 0 & 0 & cx-ay \\ 0 & 0 & 0 \end{pmatrix}$ so V_1 is a submodule

$\cong$ to the natural 2 dimensional irreducible

Similar calculation shows $\tilde{V}_1$ also irreducible, $\cong V_1^*$.

But $V_1 \cong V_1^*$.

Finally check $\left[\begin{pmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix}\right] = 0$

so V_0 is the 1-dim trivial module

Thus

$\mathfrak{sl}(3,\mathbb{C}) \cong \text{adj} \oplus \text{Natural} \oplus \text{Natural} \oplus \mathbb{C}$