Lecture 4

Review L a Lie algebra, $gl(L) = \{T: L \to L \text{ linear}\}$ is a Lie algebra under $[T_1, T_2] = T_1 \circ T_2 - T_2 \circ T_1$.

Def The adjoint homomorphism: $ad: L \to gl(L)$, $ad\,x = [x, -]: L \to L$

Def A an algebra, a derivation of A is a linear map $D: A \to A$ with

$$D(ab) = aD(b) + D(a)b \quad \forall a, b \in A.$$

Rmk For a Lie algebra L, $D: L \to L$ is a derivation if $D[x,y] = [x, Dy] + [Dx, y]$

Examples

1. $A = C^\infty(\mathbb{R})$, $Df = f'$

2. For D_1, D_2 derivations, $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is also, i.e. "Der A is a Lie subalgebra of $gl(A)$"

3. $\mathrm{Im}(ad) \subseteq \mathrm{Der}\,L \subseteq gl(L)$, i.e. $ad\,x$ is a derivation $\forall x \in L$.

   Proof
$$\begin{aligned} ad\,x([a,b]) &= [x,[a,b]] \\ &= -[a,[b,x]] - [b,[x,a]] \quad \text{by J.I.} \\ &= [a,[x,b]] + [[x,a],b] \\ &= [a, ad\,x(b)] + [ad\,x(a), b] \quad // \end{aligned}$$

Homework. Let $I\mathrm{Der}\,L \subseteq \mathrm{Der}\,L$ be all derivations of form $ad\,x$. Then these inner derivations form an ideal in $\mathrm{Der}\,L$, that is $[ad\,x, D]$ is of the form $ad(?)$ for some $? \in L$.

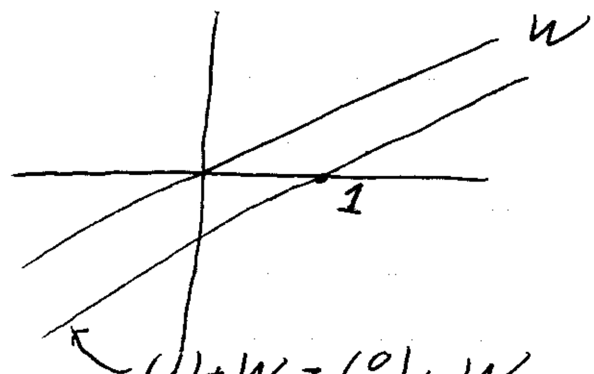
Quotient Spaces

<u>Def</u> W a subspace of V. A <u>coset</u> of W is a set of the form

$$v + W = \{ v + w \mid w \in W \}$$

<u>Important</u> $v + W = v' + W \longleftrightarrow v - v' \in W$ (cosets have many names!)

<u>Ex</u> $V = \mathbb{R}^2$ $W = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$



$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + W = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + W$
$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} + W$ etc...

<u>Def</u> Let V/W be the set of cosets of W. This is a vector space with operations:

$$(v + W) + (v' + W) = v + v' + W$$
$$\lambda (v + W) = \lambda v + W$$

<u>Rmk</u> The zero vector is $0 + W = w + W$ for any $w \in W$

<u>Rmk</u> Vector space axioms follow from those in V, need to check operation is well-defined:

<u>Ex</u> Suppose $v + W = \tilde{v} + W$, check $(v + W) + (u + W) = \tilde{v} + W + u + W$

V/W is called the <u>quotient space</u>.

<u>Prop</u> $\dim (V/W) = \dim V - \dim W$

<u>Proof</u> Choose a basis of W and extend to a basis of V.

Thm Let $T: V \to U$ be linear. Then $V/\ker T \cong \mathrm{Im}\, T$.

Proof Define $\bar{T}: V/\ker T \to \mathrm{Im}\, T$ by $\bar{T}(v+\ker) = T(v)$

- check linear (easy, T is!)
- clearly onto
- check well def: $v_1 + K = v_2 + K \leftrightarrow \cancel{T}\, v_1 - v_2 \in K$
  and 1-1 $\quad \leftrightarrow T(v_1 - v_2) = 0$
  $\quad \leftrightarrow T(v_1) = T(v_2)$ //

COR Rank-Nullity Thm

Now suppose L is a Lie algebra, I a subalgebra.
Try to define:

$$[z+L, w+L] = [z,w] + L. \quad (*)$$

Consider $l \in L$

$$[z+l+L, w+L] = [z,w] + [l,w] + L$$

Need these equal!

Fact Let L be a Lie algebra, $I \subseteq L$ an ideal. Then L/I is a <u>quotient Lie algebra</u> with bracket as in $(*)$.

Thm Let $\varphi: L_1 \to L_2$ be a Lie algebra homomorphism. Then

$$L_1/\ker\varphi \cong \mathrm{Im}\,\varphi$$ as <u>Lie algebras</u>.

Proof Check that $\bar{\varphi}$ defined above is actually a Lie algebra map

Example $L/Z(L) \cong \mathrm{IDer}\, L$

Operations with Ideals

Let I, J be ideals in L.

Trivial Fact: $I \cap J$ is an ideal.

Exercise $I + J = \{ i + j \mid i \in I, j \in J \}$ is an ideal.

Rmk This is just the subspace spanned by I & J.

Def $[I, J] = \mathrm{Span}\{ [i,j] \mid i \in I, j \in J \}$

Prop $[I,J]$ is an ideal.

Proof Let $x \in L$.

$$[x,[i,j]] = [i,[j,x]] + [j,[x,i]]$$

$$= [i, \underbrace{[j,x]}_{\in J}] - [\underbrace{[x,i]}_{\in I}, j]$$

//

Rmk Need Span, not just $\{[i,j]\}$

$$\begin{tikzcd}[column sep=small]
 & L \arrow[d, dash] & \\
 & I+J \arrow[dl, dash] \arrow[dr, dash] & \\
I \arrow[dr, dash] & & J \arrow[dl, dash] \\
 & I \cap J \arrow[d, dash] & \\
 & {[I,J]} \arrow[d, dash] & \\
 & 0 &
\end{tikzcd}$$