Lecture 27

Recall G a finite group w/ identity 1, $g \in G$. Then if $n = |G|$, $g^n = 1$.
Moreover the order of any element divides $|G|$

Def The exponent, $e(G)$ of G is the smallest positive integer such that $g^{e(G)} = 1$ $\forall g \in G$, if it exists

Rmk $e(G)$ is the lcm of orders of elts of G.

Example 1. $G = \{ z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \} = \{ e^{2\pi i/n} \mid 0 \le i < n, n \ge 1 \}$

infinite group, not finitely generated, all elts order 2

2. $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \cdots$ all elts of order 2

Rmk If G is not finitely generated, no interesting way to relate $|G|$, $e(G)$, ...

General Burnside Problem Suppose G is finitely generated and has ~~finite exp~~ all elements of finite order. Is G finite?

- G linear then Burnside showed yes ($\approx$ 1 page lin alg)
- 1964 Golod-Shafarevic: False in general

Ordinary Burnside Problem. Suppose G is fin. gen and has finite exponent. (i.e. every elt of bounded order). Must G be finite?

- 1968 Novikov-Adian — CTEX for odd exponent.
- 1992 Ivanov — CTEX for even exponent

Restricted Burnside Problem · Pretend we only care about finite groups!

Suppose G and $|G|<\infty$ is fin gen. by K elements and has exponent $e(G)=N$. Does $\exists$ a uniform constant $a(K,N)$ so $|G|\le a(K,N)$

i.e. Are there only finitely many finite groups for each K, N.

· 1956 Hall-Higman Reduce Restricted Burnside to
- p-groups
- $\approx$ finite simple groups!

2^{nd} case handled by COFSG

Thm (Zelmanov ≈ 1989, Fields Medal 1994)

Suppose G is a finite p-group with exponent p^n and generated by K elements, with $|G|=p^a$. There is a uniform bound $a(K,n)$ so that $a\le a(K,n)$

Pf
- Construct a Lie alg in characteristic p from lower central series of G
- Show this Lie alg is nilpotent $\Rightarrow$ RBP
- Prove ∞-dimensional version of Engel's Thm to handle this

First Case $p=2$, $n=1$ so $e(G)=2$, i.e. $g^2=1\ \forall g\in G$.

Exercise Suppose G is k-generated and $e(G)=2$. Then G is abelian and $|G|\leq 2^k$.

Next Case $p^n=3$,

Some group Theory

Def $x,y\in G$ define $[x,y]=x^{-1}y^{-1}xy$ "commutator of x & y"

Rmk $[x,y]=1$ iff x,y commute.

Def $H,K\leq G$ then $[H,K]=$ subgroup gen by $\{[h,k]\mid h\in H, k\in K\}$

Def $G'=[G,G]$ commutator subgroup (compare to derived alg)

Rmk G/G' abelian, $H\trianglelefteq G$, G/H abelian $\Leftrightarrow G'\leq H$.

Def $G=G^0$, $G^1=[G,G]$, $G^2=[G,G^1]$, .., $G^i=[G,G^{i-1}]$

$G=G^0\geq G^1\geq\ldots$ is the lower central series

Def G is nilpotent if $G^m=1$ for some m

Rmk 1. G^i/G^{i+1} is abelian and f.g. if G is

2. If $e(G)=p$ then G^i/G^{i+1} is elementary abelian, $\mathbb{Z}/p\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}/p\mathbb{Z}$, so is a vector space $/\mathbb{F}_p$

n=Lcase

$(ab)^3 = 1 \Rightarrow aba = b^{-1}a^{-1}b^{-1}$ also $g^2 = g^{-1}$, $h^2 = h^{-1}$

$$
\begin{aligned}
[g,[g,h]] &= g^{-1}[g,h]^{-1}g[g,h] \\
&= g^{-1}h^{-1}g^{-1}hggg^{-1}h^{-1}gh \\
&= g^{-1}h^{-1}(g^{-1}hg^{-1})\underline{gg}\,h^{-1}gh \\
&= g^{-1}\underline{h^{-1}h^{-1}}gh^{-1}\underline{g^{-1}}\,h^{-1}gh \\
&= g^{-1}hg(h^{-1}g^{-1}h^{-1})gh \\
&= g^{-1}h\underline{g\,g}h\underline{g\,g}h \\
&= (g^{-1}hg^{-1})\,hg^{-1}h \\
&= \cancel{g}h^{-1}gh^{-1}hg^{-1}h \quad = \underline{1}
\end{aligned}
$$

__Rmk__ We proved ad g is nilpotent.
Exc 4.8 says this suffices (since all $(\mathrm{ad}\, g)^2 = 0$)

General case needs ∞-dim'l Engel.