Lecture 26

Review L a Lie algebra. A representation is a Lie alg homomorphism
$\varphi: L \to gl(V) = gl(n, F)$. Say V is an L-module:
$\varphi(x)(v) = x \cdot v$ and $[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$

Irreducible $sl(2,\mathbb{C})$ modules
- V_d of dimension $d+1$, one for each $d \geq 0$
- eigenvalues of h are $d, d-2, d-4, \ldots, -d$
- e and f move e-vectors around, "highest weight vector" has
  $h \cdot v = dv$, $e \cdot v = 0$.

Weyl's Thm L a s.s. Lie alg $/\mathbb{C}$. Any finite-dimensional L-module is a $\oplus$ of irreducibles.

Goal For each semisimple Lie alg L, obtain a classification of irreducible L-modules like we did for $sl(2,\mathbb{C})$

Step 1 Choose a CSA H, root system Φ and base $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$.
Let Φ^+, Φ^- be the positive and negative roots. Then
$\exists$ a triangular decomposition

$$L = N^- \oplus H \oplus N^+ \qquad N^- = \bigoplus_{\alpha \in \Phi^-} L_\alpha, \; N^+ = \bigoplus_{\alpha \in \Phi^+} L_\alpha.$$

Rmk 1. For $sl(2,\mathbb{C})$ $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $L_{\alpha_1} = \langle e \rangle$, $L_{-\alpha_1} = \langle f \rangle$

$$L = \langle f \rangle \oplus \langle h \rangle \oplus \langle e \rangle$$

2. Note that N^- and N^+ are subalgebras

3 For $sl(n,\mathbb{C})$ N^+ is strictly upper Δ N^- str. lower Δ.

Step 2 Just as in $sl(2,\mathbb{C})$ case, consider action of H.
So let V be a f.d. L-module, V has a basis of simultaneous e-vectors. Thus

Def For $\lambda \in H^*$, $V_\lambda = \{ v \in V \mid h \cdot v = \lambda(h) v \ \forall h \in H \}$
$\Psi = \{ \lambda \in H^* \mid V_\lambda \neq 0 \}$. Then

$$V = \bigoplus_{\lambda \in \Psi} V_\lambda$$ is the <u>weightspace decomposition</u>.

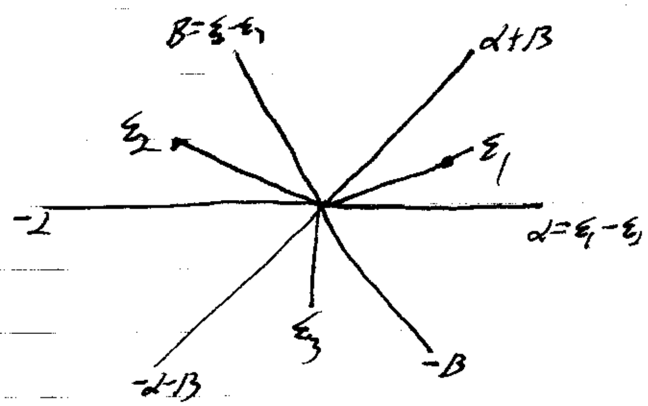
Ex. $V = L$, adjoint representation. Then weights = roots, ~~&~~ wt space decomp is root space.

Ex. $L = sl(3,\mathbb{C})$, $V = \mathbb{C}^3$ natural rep.

$$\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} h_1 a_1 \\ h_2 a_2 \\ h_3 a_3 \end{pmatrix}$$

<u>weight λ</u>	<u>V_λ</u>
ε_1	$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$
ε_2	$\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$
ε_3	$\left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

Project H^* onto plane:



$\varepsilon_1 = \frac{1}{3}(2\alpha + \beta)$ on H
since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$

weights of natural rep and adjoint rep of $sl(3,\mathbb{C})$

Step 3 Consider action of $e_\alpha, f_\alpha : \alpha \in \Phi$ on V.

Ex $e_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$ $\quad f_\alpha \cdot V_\lambda \in V_{\lambda-\alpha}$

Cor/Def $\exists$ some $\lambda \in \Psi$ so $\forall \alpha \in \Phi^+$, $\alpha+\lambda \notin \Psi$. Say λ is a highest weight and $0 \neq v \in V_\lambda$ is a highest weight vector.

Ex Nat Rep on $\mathfrak{sl}(3,\mathbb{C})$ $\quad \Phi^+ = \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_1-\varepsilon_3$

Highest weight is ε_1.

Lemma Let V be an irreducible L-module, Ψ the set of weights. Then Ψ has a unique highest weight λ, V_λ is one-dimensional, and all other weights are of form:

$$\lambda - \sum_{\alpha_i \in \Pi} a_i \alpha_i, \quad a_i \in \mathbb{Z}^{\geq 0}.$$

Pf Mimic $\mathfrak{sl}_2(\mathbb{C})$ proof using all the e_α's and f_α's.

Ex $L = \mathfrak{sl}(\ell+1, \mathbb{C})$, $V = L$. Highest weight is $\alpha_1+\alpha_2+\cdots+\alpha_\ell = \varepsilon_1 - \varepsilon_\ell$

Highest weight vector $e_{1,\ell+1}$.

Rmk λ a weight, $\alpha \in \Pi$ then $\lambda(h_\alpha) \in \mathbb{Z}$. Suppose $\lambda(h_\alpha) < 0$, then by $\mathfrak{sl}(2,\mathbb{C})$ theory $e_\alpha \cdot L_\lambda \neq 0 \Rightarrow \alpha + \lambda \in \Psi$.

Cor λ a highest weight $\Rightarrow \lambda(h_\alpha) \geq 0 \ \forall \alpha \in \Pi$.

Thm Let $\Lambda = \{\lambda \in H^* \mid \lambda(h_\alpha) \in \mathbb{Z}$ and $\lambda(h_\alpha) \geq 0 \quad \forall \alpha \in \Pi\}$.
For each $\lambda \in \Lambda$, $\exists$ a finite-dimensional simple L-module, denoted $V(\lambda)$, w/ highest wt λ

Moreover any two simple modules w/ same highest wt are $\cong$ and every simple L-module occurs this way

Def Λ called set of <u>dominant weights</u>

Simple f.d. L-modules $\longleftrightarrow$ Dominant weights

Ex $L = \mathfrak{sl}(2, \mathbb{C}) \qquad \lambda_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \qquad \Lambda = \mathbb{Z}^+ \cdot \lambda_1$

Def $\exists!$ elements $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in H^*$ such that

$$\lambda_i(h_{\alpha_j}) = \delta_{ij}$$ called <u>fundamental dominant weights</u>.

COR $\Lambda = \mathbb{Z}^{\geq 0} \cdot$ FDWts

Exc $\lambda_i = \sum_{k=1}^{\ell} d_{ik} \alpha_k \rightsquigarrow d_{ik} = C^{-1}$

$L = \mathfrak{sl}(3, \mathbb{C})$

$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) = \frac{1}{3}(2\varepsilon_1 - 2\varepsilon_2 + \varepsilon_2 - \varepsilon_3)$$
$$= 1/3(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) = \varepsilon_1$$
$$\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) = \frac{1}{3}(\varepsilon_1 - \varepsilon_2 + 2\varepsilon_2 - 2\varepsilon_3)$$
$$= \frac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) = -\varepsilon_3$$

Irreducible $sl(3,\mathbb{C})$ modules

Cor For any pair of natural #'s a, b $\exists!$ irreducible $sl(3,\mathbb{C})$ module w/ highest weight $a\varepsilon_1 - b\varepsilon_3$, call it ~~$\Gamma_{a,b}$~~ $V(a,b)$

Ex Natural $\cong V(1,0)$
adjoint $\cong V(1,1)$ 8-dimensional.

Rmk To construct the rest use tensor products of symmetric powers etc..

Ex $V \otimes V(2,1) \cong V(3,1) + V(1,2) + V(2,0)$

$Sym^2 V \otimes V^* \cong V(2,1) \oplus V$ etc.,

Weyl Character Formula

- Combinatorial Description of weight space multiplicities in $V(\lambda)$.

Ex $\dim(V(a,b)) = \dfrac{(a+b+2)(a+1)(b+1)}{2}$