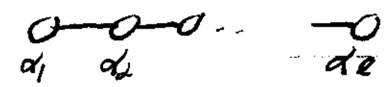
Lecture 25

Review. Last class we constructed the classical Lie algebras and proved they are simple by explicitly finding root systems / Dynkin Diagrams.

Type A_ℓ $\quad L = \mathfrak{sl}(\ell+1, \mathbb{C}) \qquad \Phi = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le \ell\}$

$\dim L = (\ell+1)^2 - 1 = \ell^2 + 2\ell$

Base $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$



Type B_ℓ $\quad L = \mathfrak{so}(2\ell+1, \mathbb{C}) = \mathfrak{gl}_S(2\ell+1, \mathbb{C})$ for $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$

$$L = \left\{ \begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix} \;\middle|\; p = -p^t,\ q = -q^t \right\}$$

$\Phi = \{\pm\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le \ell\}$

$\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \beta_\ell = \varepsilon_\ell\}$

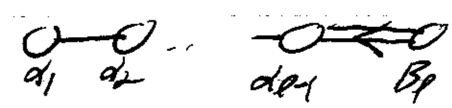
$\dim L = 2\ell^2 + \ell$

Type C_ℓ $\quad L = \mathfrak{sp}(2\ell, \mathbb{C}) = \mathfrak{gl}_S(2\ell, \mathbb{C}) \qquad S = \left\{ \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} \right\}$

$L = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} \;\middle|\; p = p^t,\ q = q^t \right\} \qquad \dim L = 2\ell^2 + \ell$

$\Phi = \{\varepsilon_i - \varepsilon_j\ i \ne j\} \cup \{\pm(\varepsilon_i + \varepsilon_j)\ i<j\} \cup \{\pm 2\varepsilon_i\}$

$\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \beta_\ell = 2\varepsilon_\ell\}$

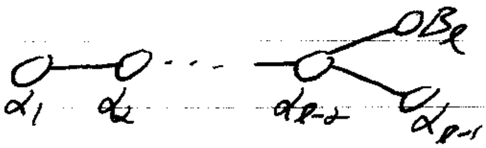


<u>Type D_ℓ</u> $L = so(2\ell, \mathbb{C}) = gl_S(2\ell, \mathbb{C})$ $\quad S = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$

$L = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} \mid p = -p^t, q = -q^t \right\}$ $\quad \dim L = 2\ell^2 - \ell$

$\Phi = \{ \varepsilon_i - \varepsilon_j \ \ i \neq j \} \cup \{ \pm(\varepsilon_i + \varepsilon_j) \ \ i \neq j \}$

$B = \{ \alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \beta_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell \}$



<u>Thm</u> Let L be a $\mathbb{C}$ semisimple Lie alg. If Φ_1, Φ_2 are root systems for different Cartan's, they are $\cong$.

<u>Pf</u> App C

<u>COR</u> Only $\cong$'s btw Classical Lie algs are from identical Dynkin diag

<u>Ex</u> $D_3 = A_3$, one can show $so(6, \mathbb{C}) \cong sl(4, \mathbb{C})$.

<u>Need</u> Given Φ irred there is a <u>unique</u> simple Lie alg.

<u>Rmk</u> $(X,Y) = tr(XY)$ always nondeg so $\kappa(X,Y) = ?\, tr(XY)$ in each case.

Serre's Thm

Setup L a $\mathbb{C}$ s.s. Lie alg w/ Cartan H root system Φ, base $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$.
Let $sl(\alpha_i) = \langle e_i, f_i, h_i \rangle$.

Goal $\{e_i, f_i, h_i \mid 1 \le i \le \ell\}$ generate L with relations depending only on Cartan matrix.

Lemma 1 Notation as above, then L is generated as a Lie alg by $\{e_i, f_i \mid 1 \le i \le \ell\}$.

Pf Note $h_i = [e_i, f_i]$. Since $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ span H^* one easily sees $\{h_i\}$ span H. Thus ETS each $L_\beta : \beta \in \Phi$ can be obtained.

We know $\beta = w(\alpha_j)$ some $w \in W$ a product of simple reflections.
Let $\tilde{L}$ = subalg (gen by e_i, f_i). ETS.

$$L_\gamma \in \tilde{L},\ \beta = s_{\alpha_i}(\gamma) \Rightarrow L_\beta \in \tilde{L}.$$

So $\beta = \gamma - \langle \gamma, \alpha_i \rangle \alpha_i$ but $\bigoplus_k L_{\gamma + k\alpha_i}$ is irred $sl(\alpha_i)$ module. So applying $[e_i, -]$ or $[f_i, -]$ we get $L_\beta \in \tilde{L}$. //

Notation $c_{ij} = \langle \alpha_i, \alpha_j \rangle \le 0$ Cartan Matrix.

Lemma 2 (Serre Relations) The elements $\{e_i, f_i, h_i\}$ satisfy:

(S1) $[h_i, h_j] = 0 \quad \forall i, j$
(S2) $[h_i, e_j] = c_{ji}\, e_j \quad [h_i, f_j] = -c_{ji}\, f_j \quad \forall i, j$
(S3) $[e_i, f_j] = \delta_{ij}\, h_i$

(S4) $(\operatorname{ad} e_i)^{1 - c_{ji}}(e_j) = 0 \quad (\operatorname{ad} f_i)^{1 - c_{ji}}(f_j) = 0 \quad$ if $i \ne j$

Proof

(S1) Holds as H is a CSA

(S2) $[h_i, e_j] = \alpha_j(h_i) e_j = \langle \alpha_j, \alpha_i \rangle e_j$ (see Ex 10.4)

(S3) $[e_i, f_i] = h_i$ from $sl(2,\mathbb{C})$. $[e_i, f_j] \in L_{\alpha_i - \alpha_j} = 0$
since $\alpha_i - \alpha_j \notin \Phi$.

(S4) Fix α_i, α_j consider $M = \bigoplus_K L_{\alpha_j + K\alpha_i}$, all K w/ $\alpha_j + K\alpha_i \in \Phi$.

This is an $sl(\alpha_i)$-module.

Note For $x \in L_{\alpha_j + K\alpha_i}$, $[h_i, x] = (\alpha_j(h_i) + K\alpha_i(h_i))x$
$= (\langle \alpha_j, \alpha_i \rangle + 2K)x$

Note $K=0$ occurs $K=-1$ does not.

* $c_{ji} = \langle \alpha_j, \alpha_i \rangle$ is smallest e-value of $\mathrm{ad}\, h_i$ on M
Thus $-c_{ji}$ is largest e-value of $\mathrm{ad}\, h_i$ on M

But $x = (\mathrm{ad}\, e_i)^{-c_{ji}}(e_j)$ is such an e-vector, so $K = -c_{ji}$
it is annihilated by $\mathrm{ad}\, e_i$.

Similar for 2nd part. //

Serre's Thm Let C be a Cartan matrix of a root system. Define a complex Lie algebra L generated by $\{e_i, f_i, h_i\}$ subject to relations (S1) to (S4). Then

- L is fin dim & semisimple
- L has a Cartan subalg H spanned by $\{h_1, h_2, \ldots, h_\ell\}$
- Root system has Cartan matrix C.

COR Exceptional Lie algebras exist.

COR We have classified simple $\mathbb{C}$-Lie algebras!

Pf See Humphreys

Long! Start w/ ∞-dimensional just satisfying S1-3.