Lecture 23

Review: $L$ s.s. Lie alg $\Rightarrow L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \Rightarrow$ abst root system $R$ w/ base $B \Rightarrow$ Cartan Matrix

**Def** $R$ w/ base $\{\alpha_1, \alpha_2, ..., \alpha_\ell\}$, $C_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$

- $C_{ii} = 2$, $C_{ij} \leq 0$ $i \neq j$
- Can recover $R$ from $C$

**Def** Dynkin Diagram $\Delta(R)$ w/ vertices $\alpha_i$ and $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle = d_{ij}$ edges connecting $\alpha_i, \alpha_j$. Orient edges $\beta \to \alpha$ if connected and $(\beta, \beta) > (\alpha, \alpha)$.

- $\Delta(R)$ determines $R$ up to $\cong$.

**Prob** Which Dynkin diagrams occur as $\Delta(R)$? We first ignore $\to$ ~~and multedges~~

**Def** Let $E$ be an $\mathbb{R}$-vector space w/ inner product $(,)$. $A \subseteq E$ is admissible if $A = \{v_1, v_2, ..., v_n\}$ is lin. ind. and

1. $(v_i, v_i) = 1$ $\forall i$, $(v_i, v_j) \leq 0$ $\forall i \neq j$
2. $i \neq j \Rightarrow 4(v_i, v_j)^2 = \{0, 1, 2, 3\}$

**Def** Graph $\Gamma_A$ w/ vertices $v_i$ and $4(v_i, v_j)^2$ edges conn. $v_i \to v_j$.

**Ex** $B$ a base of $R$, $A = \{\frac{\alpha_i}{|\alpha_i|} \mid \alpha_i \in B\}$ is admissible and $\Gamma_A$ is $\Delta(R)$ w/out the arrows, a.k.a. the Coxeter graph.

ASSUME $\Gamma_A$ CONN.

**Lemma 1** $A$ admissible $\Rightarrow$ any subset is.

**Lemma 2** At most $|A| - 1$ pairs of vertices are joined by an edge.

**Pf** Let $A = \{v_1, v_2, ..., v_n\}$ and $0 \neq v = v_1 + v_2 + \cdots + v_n$.

Then $0 < (v,v) = n + \sum_{i<j} 2(v_i,v_j) = n + \sum_{i<j} -\sqrt{d_{ij}}$ so

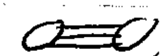
$$n > \sum_{i<j} \sqrt{d_{ij}} \geq \text{Total \# pairs connected. //}$$

**COR 1** $\Gamma_A$ has no cycles.

Pf: Cycle is admissible by Lemma 1 but too many edges for Lemma 2.

**Lemma 3** No vertex is connected to $\geq 4$ edges.

Pf If $v$ conn. to $v_1, v_2, \dots, v_k$ then all $(v_i, v_j) = 0$ since no triangles. Use that every nonzero $(v, v_j)$ is at least $1/4$ but $(v,v) = 1$ //

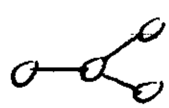
**COR 2** If $\Gamma_A$ is connected and has a triple edge then $\Gamma_A =$ 

**Shrinking Lemma** Suppose  $\in \Gamma$. Let $v = \sum_{i=1}^{k} v_i$

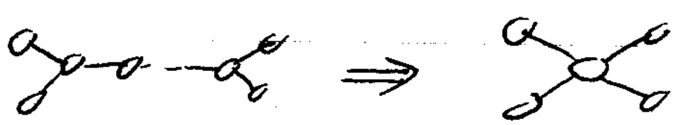
Replacing $\{v_1, \dots, v_k\}$ with $\{v\}$ in $A$ gives an admissible set $A'$ where $\Gamma_{A'}$ is $\Gamma_A$ with $v_1, \dots, v_k$ compressed to one vertex.

Pf Exercise

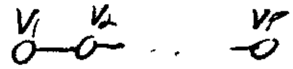
**Lemma 4** $\Gamma_A$ has

1. $\leq 1$ double edge
2. $\leq 1$ branch point: 
3. Not Both

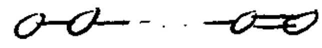
**Proof** 1. Suppose , use shrinking lemma to get  $\nexists$

2. Again  $\nexists$

3. As above //

**Lemma 5** Suppose $\Gamma$ has  . Let $V = \sum_{i=1}^{p} i V_i$. Then
$$(V,V) = \frac{p(p+1)}{2}$$

**Pf** We know $2(V_i, V_{i+1}) = -1$, rest are $0$. Thus
$$(V,V) = \sum_{i=1}^{p} i^2 + 2\sum_{i=1}^{p-1} (V_i, V_{i+1})\, i(i+1) = p(p+1)/2 \quad //$$

**Prop** Suppose $\Gamma$ has a double edge. Then $\Gamma =$  or .

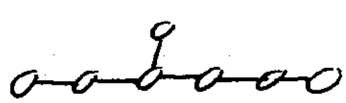
**Pf** By Lemma 4 $\Gamma =$  $p \geq q$ WLOG.

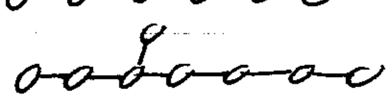
Set $V = \sum_{i=1}^{p} i V_i$ and $W = \sum_{i=1}^{q} i W_i$, so $(V,V) = \frac{p(p+1)}{2}$, $(W,W) = \frac{q(q+1)}{2}$.

Graph gives $4(V_p, W_q)^2 = 2$ and all other $(V_i, W_j) = 0$. Then
$$(V,W)^2 = (pV_p, qW_q)^2 = \frac{p^2 q^2}{2}. \text{ But } V, W \text{ lin ind so}$$
$$(V,W)^2 < (V,V)(W,W) \Rightarrow 2pq < (p+1)(q+1)$$
$$\Rightarrow (p-1)(q-1) < 2 \text{ so } q = 1 \text{ or } p = q = 2 //$$

**Prop** Suppose $\Gamma$ has a branch point. Then $\Gamma$ is one of

$D_n$  , $E_6$ 

$E_7$ 

$E_8$ 

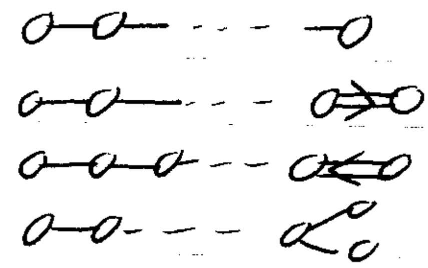
**Thm** Let $R$ be an irreducible root system. Then the Dynkin diagram is one of four families:

$A_\ell$ $\ell \geq 1$

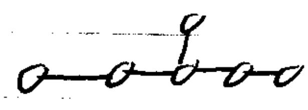
$B_\ell$ $\ell \geq 2$

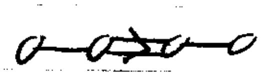
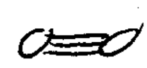
$C_\ell$ $\ell \geq 3$

$D_\ell$ $\ell \geq 4$



or one of five exceptional diagrams

$E_6$  $E_7$  $E_8$ 

$F_4$  $G_2$ 

**Next Class** Types A–D arise from classical Lie algs

**Type $G_2$**

$$E = \{ v = \sum_{i=1}^{3} c_i \varepsilon_i \in \mathbb{R}^3 \mid \sum c_i = 0 \}$$

$$I = \{ m_1\varepsilon_1 + m_2\varepsilon_2 + m_3\varepsilon_3 \in \mathbb{R}^3 \mid m_i \in \mathbb{Z} \}$$

$$R = \{ \alpha \in I \cap E \mid (\alpha,\alpha) = 2 \text{ or } (\alpha,\alpha) = 6 \}$$

$$= \{ \pm(\varepsilon_i - \varepsilon_j) \mid i \neq j \} \cup \{ \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \} \quad 12 \text{ roots}$$

$$\alpha = \varepsilon_1 - \varepsilon_2, \quad \beta = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 \quad \text{base,}$$

## Type $E_8$ $\quad E = \mathbb{R}^8$

$$R = \{\pm\varepsilon_i \pm \varepsilon_j : i<j\} \cup \{\tfrac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i \mid \text{even \# of plusses}\}$$

$\binom{8}{2}\cdot 2\cdot 2 = 112$ roots

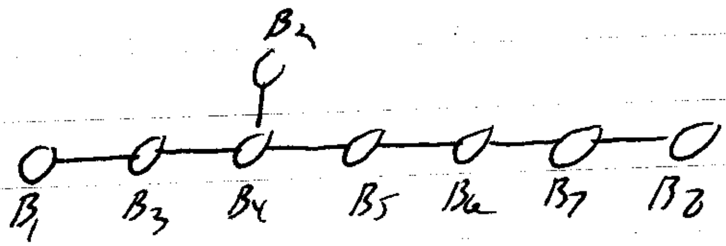
$\binom{8}{0}+\binom{8}{2}+\binom{8}{4}+\binom{8}{6}+\binom{8}{8} = 128$

240 roots

Base: $\beta_1 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_8 + \sum_{i=2}^{7} \varepsilon_i)$

$\beta_2 = -\varepsilon_1 - \varepsilon_2$

$\beta_i = \varepsilon_{i-2} - \varepsilon_{i-1} \quad 3 \le i \le 8$




$E_7$ omit $\beta_8$

$E_6$ omit $\beta_8, \beta_7$

$\therefore \dim L = \dim H + |\Phi|$

$= 8 + 240 = 248$