Lecture 21

11.2, 11.5, 11.6, 11.10, 11.13

Review E a f.d. vectorspace over $\mathbb{R}$ w/ inner product $(\,,)$. For $0 \neq v \in E$:

$$s_v(x) = x - \langle x,v \rangle v$$

is reflection across hyperplane $\perp$ to v, where $\langle x,v \rangle = \frac{2(x,v)}{(v,v)}$.

Def $R \subseteq E$ is a root system if

1. R is finite, spans E and $0 \notin R$.
2. $\alpha \in R$ then $c\alpha \in R \leftrightarrow c = \pm 1$.
3. $\alpha \in R$ then s_α permutes R
4. $\forall \alpha, \beta \in R \quad \langle \beta, \alpha \rangle \in \mathbb{Z}$.

* Changing $(\,,)$ by scalar changes nothing

Main Example L a complex s.s. Lie algebra, $R = \Phi \subseteq \mathbb{R}\text{span}\,\Phi = E \subset H^*$.

Ex Consider $\mathbb{R}^{\ell+1}$ w/ usual inner product. $R = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq \ell+1\}$

Let $E = \langle R \rangle$ ℓ-dimensional subspace.

* This is $\cong$ to the root system for $\mathfrak{sl}(\ell+1, \mathbb{C})$ $e_i \leftrightarrow \varepsilon_i \in H^*$.

Finiteness Lemma $R \subset E$ a root system, $\alpha, \beta \in R$ with $\alpha \neq \pm\beta$. Then

$$\langle \alpha,\beta \rangle \langle \beta,\alpha \rangle \in \{0,1,2,3\}$$

Proof Recall "$x \cdot y = |x||y|\cos\theta$", so $(\alpha,\beta)^2 = (\alpha,\alpha)(\beta,\beta)\cos^2\theta$

so $$4\frac{(\alpha,\beta)(\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta)} = \langle \alpha,\beta \rangle \langle \beta,\alpha \rangle \leq 4\cos^2\theta \leq 4$$

But $= 4$ means $\cos\theta = \pm 1 \Rightarrow \alpha = \pm\beta$ $\divideontimes$ //

Rmk $\langle \alpha,\beta \rangle, \langle \beta,\alpha \rangle$ always same sign, ≤ 0 if obtuse $\measuredangle$, ≥ 0 if acute

<u>Rmk</u> The Finiteness Lemma severely restricts ratios of lengths and ∡s btw roots $\in R$

Let $\alpha, \beta \in R$, $\alpha \neq \pm\beta$.

* Assume WLOG that $(\beta,\beta) \geq (\alpha,\alpha)$ so $|\langle\beta,\alpha\rangle| \geq |\langle\alpha,\beta\rangle|$

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\frac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\pi/2$	not determined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

<u>Prop</u> Let $\alpha, \beta \in R$.

(a) If ∡ btw them is strictly $> \pi/2$ then $\alpha+\beta \in R$.

(b) If ∡ btw them is $< \pi/2$, then $\alpha-\beta \in R$.

and $(\beta,\beta) \geq (\alpha,\alpha)$

<u>Proof</u> WLOG assume $(\beta,\beta) \geq (\alpha,\alpha)$ (case (a) is symmetric)

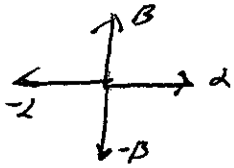
$$S_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta \in R$$ by hypothesis, use table //

<u>Root Systems in $\mathbb{R}^2$</u>

Let $R \subset \mathbb{R}^2$ be a root system. Choose $\alpha \in R$ of minimal length. Choose $\beta \neq \pm\alpha$. WLOG β makes ∡ $\geq \pi/2$ w/ α.

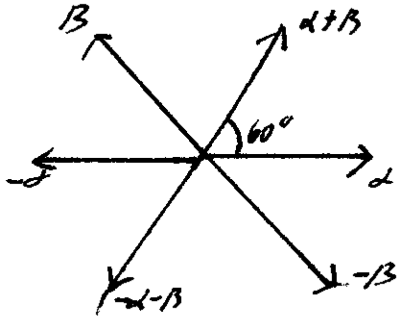
So choose β so its angle with α is as large as possible

Case 1 $\theta = \pi/2$.



Type $A_1 \times A_1$. – Note (α,α), (β,β) are not related

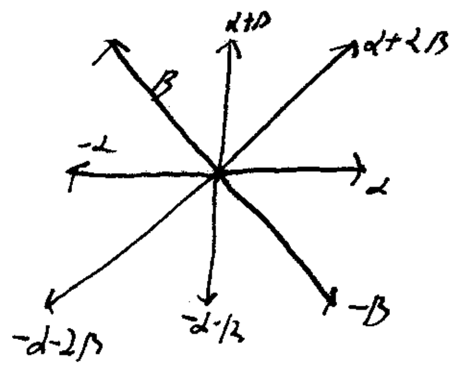
Case 2 $\theta = 2\pi/3$. By the table, α & β have same length and $\alpha+\beta \in R$.



Type A_2

Note any more roots would give an ∢ bigger than $2\pi/3$

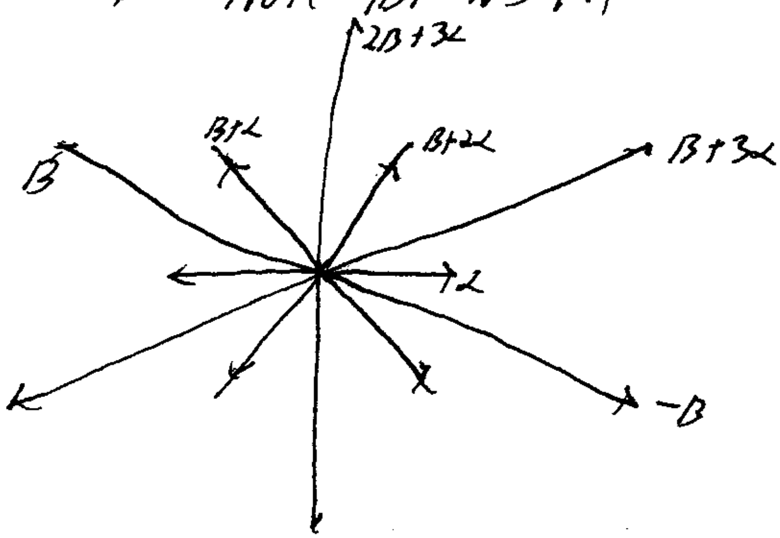
Case 3 $\theta = 3\pi/4$, Note $|\beta|^2 = 2|\alpha|^2$ so $|\beta| = \sqrt{2}\,|\alpha|$, also $\alpha+\beta$ is a root.



Apply s_α to β to get $\alpha+2\beta \in R$.

Type B_2

Case 4 $\theta = 5\pi/6$ Note $|\beta| = \sqrt{3}\,|\alpha|$



Type G_2

Def R is <u>irreducible</u> if $\nexists$ disjoint subsets $R_1, R_2 \subset R$ with $R = R_1 \cup R_2$ and $(\alpha, \beta) = 0$ $\forall \alpha \in R_1, \beta \in R_2$.

Prop Any root system is a disjoint union of irreducible root systems $R = R_1 \cup \cdots \cup R_s$ with $\text{Span}\{R_i\} = E_i$ and $E_i \perp E_j$ $\forall i \neq j$

Pf Exercise: Partition R as finely as possible into mutually $\perp$ subsets.

Bases

Def A subset B of R is a <u>base</u> if
1. B is a basis of E
2. Every $\beta \in R$ is of form $\beta = \sum_{\alpha \in B} c_\alpha \cdot \alpha$ with $c_\alpha \in \mathbb{Z}$

   and all $\neq 0$ c_α of the same sign.

Def With respect to a base B #2 partitions the roots into <u>positive roots</u> and <u>negative roots</u>. Elements in B are called <u>simple roots</u> and $\{s_\beta \mid \beta \in B\}$ are <u>simple reflections</u>.

Ex

$\mathfrak{sl}(3, \mathbb{C})$ root system. Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$.
$B = \{\alpha_1, \alpha_2\}$ is a base.

Pos Roots $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$
Neg Roots $\Phi^- = \{-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$

Ex $\mathfrak{sl}(n, \mathbb{C})$, generalize above.

Rmk All this depends on choice of B.

Thm Every Root System Has a Base

Pf • Choose $z \in E$ not $\perp$ to any root
• Let $R^+ = \{\alpha \in R \mid (z,\alpha) > 0\}$
• Set $B = \{\alpha \in R^+ \mid \alpha$ is not a sum of two other roots$\}$

Check it all works!

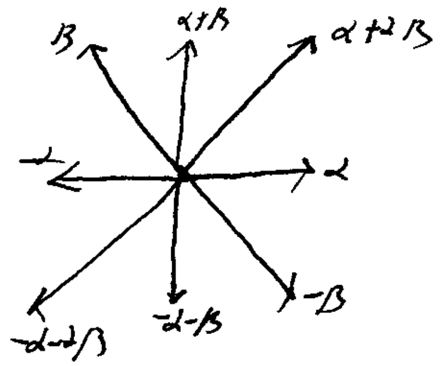
Weyl Group

Def Let R be a root system, so $\{s_\alpha \mid \alpha \in R\}$ are all in $GL(E)$.
The Weyl group of R, denoted W or $W(R)$, is the subgroup of $GL(E)$ generated by $\{s_\alpha \mid \alpha \in R\}$.

Prop W is finite.

Pf We know each s_α permutes R so this gives a group homom. $W \to Sym(R)$
If $g \in$ Kernel then g fixes all roots $\Rightarrow g = Id$ since they span

Ex $W(B_2)$



Check
• Generated by s_α, s_β
• $s_\alpha \circ s_\beta$ is a 90° clockwise rotation

Thus $W(B_2) = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = e,\ (s_\alpha s_\beta)^4 = e \rangle$

$\cong D_8$