Lecture 2

Review
- Field F
- V a vector space over F
- A an F algebra (may be assoc, commut, unital)

Def A *basis* of V is a subset of V which is linearly independent and spans V.

Thm Any vector space has a basis. The cardinality of a basis is independent of choice and called the *dimension* of V.

Rmk Requires axiom of choice in ∞ dim'l setting.

Examples

1. F^n, std basis $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

2. $V = \{2\times 2 \text{ matrices w/ trace zero}\}$ $\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

3. $F[x]$ basis $\{1, x, x^2, x^3, \ldots\}$

Linear Maps

Def Let V, W be vector spaces $/F$. $T: V \to W$ is *linear* if
$$T(v_1+v_2) = T(v_1) + T(v_2)$$
$$T(\lambda v) = \lambda T(v)$$
$$\forall v_1, v_2 \in V,\ \lambda \in F$$

Exercise
1. $\text{Kernel } T = \{v \in V \mid T(v) = 0\}$ is a subspace of V (aka nullspace)
2. $\text{Image } T = \{T(v)\}$ is a subspace of W

Rank-Nullity Thm Let $T: V \to W$ be linear.

$$\dim V = \underbrace{\dim(\text{Image } T)}_{\text{Rank}} + \underbrace{\dim(\text{Ker } T)}_{\text{Nullity}}$$

<u>Def</u> If $T: V \to W$ is 1-1 and onto, say T is an $\cong$ and $V \cong W$.

<u>Key Fact</u> Let V have basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \mathcal{B}$. Given $\vec{v} \in V$ let $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$. The map

$$\vec{v} \to [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ gives an $\cong$ from V to F^n.

* Every n-dimensional v.s /F is $\cong$ to F^n *

<u>Examples</u>

1. V as in #2 above
2. $P_n[x]$

<u>Lie Algebras</u>

<u>Def</u> A <u>Lie algebra</u> over F is an F-vector space together with a map $[\,,\,]: L \times L \to L$, <u>the Lie bracket</u>, such that

0. $[\,,\,]$ is <u>bilinear</u>
$$[x+y, z] = [x,z] + [y,z]$$
$$[x, y+z] = [x,y] + [x,z]$$
$$\lambda[x,y] = [\lambda x, y] = [x, \lambda y]$$
Think: Linear in each coordinate

(L1) $[x,x] = 0 \quad \forall x \in L$

(L2) Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall x,y,z \in L$

<u>Rmk</u> Expand $0 = [x+y, x+y]$ and use L(1) to get

(L1') $[x,y] = -[y,x] \quad \forall x,y \in L$

Assume: L is finite-dimensional (except at very end)

Rmk The Jacobi identity should seem mysterious and not so natural!

Examples.

1. Let L be any vector space and define $[x,y]=0$ $\forall x,y \in L$. This is called the abelian Lie algebra structure on L.

2. Pick V a finite dimensional v.s. $/F$. Define:

$$\mathfrak{gl}(V) = \{x : V \to V \text{ linear}\}$$ - note this is a vector space

Def $[x,y] = x \circ y - y \circ x \in \mathfrak{gl}(V)$

Bilinear ✓, (L1) ✓

Jacobi
$$\begin{aligned} & x\circ(y\circ z - z\circ y) - (y\circ z - z \circ y)\circ x \\ + \; & y \circ (z\circ x - x \circ z) - (z\circ x - x\circ z)\circ y \\ + \; & z \circ (x \circ y - y \circ x) - (x\circ y - y \circ x)\circ z \\ \hline & = 0 \;!! \end{aligned}$$

3 Recall: Choosing a basis for V, every linear map can be written as an $n\times n$ matrix, matrix multiplication corresponds to composition.

Def $\mathfrak{gl}(n,F) = \{n\times n \text{ matrices } /F\}$
$[A,B] = AB - BA$. general linear subalgebra

Basis
$$e_{ij} = \begin{array}{c c} & j \\ i & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \end{array}$$

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

4. Via structure constants. Suppose Basis $\{x_1, x_2, \dots, x_n\}$
Set:

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$$

Rmk 1 Very hard to check Jacobi identity
2 Lie algebras $a_{ii}^k = 0$, $a_{ij}^k = -a_{ji}^k$.

5. $sl_n(F) = \{A \in gl_n(F) \mid \text{trace } A = 0\}$ is a Lie subalgebra of $gl(n, F)$

$*$ $tr(AB) = tr(BA)$ so $tr(AB - BA) = 0$.

Called special linear algebra

~~5.5 Heisenberg = strictly upper~~ Δ 3x3 matrices

6. $b(n, F)$ = upper triangular matrices in $gl(n, F)$

7. $L = \mathbb{R}^3$, $[x, y] = x \wedge y$ (aka $x \times y$ cross product)
~~$(x_1, y_1, z_1) = (x_2, y_2, z$~~ $(x, y, z) \times (a, b, c) = (yc - zb, za - xc, xb - ya)$
Annoying: Check Jacobi
MTH 241: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, $\vec{u} \times \vec{u} = \vec{0}$.

Subalgebras and Ideals

Def. Suppose L is a Lie algebra. A Lie subalgebra is a subspace K which is also a Lie alg under same bracket, i.e.

$$[x, y] \in K \quad \forall x, y \in K.$$

Def. An ideal is a subspace I so $[x, y] \in I$ $\forall x \in L, y \in I$.

Obvious: ideals are subalgebras

Examples

1. $\{0\}$ and L are always ideals.

2. The center $Z(L) = \{ x \in L \mid [x,y]=0 \ \forall y \in L \}$

   Hint: Use Jacobi to prove ideal

3. $b_n(F)$ is a subalgebra but not an ideal of $gl(n,F)$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \notin b_2(F)$$

Homomorphisms

Let L_1, L_2 be two Lie algebras over F. A Lie alg. homomorphism is a linear map $\varphi: L_1 \to L_2$ such that
$$\varphi[x,y] = [\varphi(x), \varphi(y)] \quad \forall x,y \in L_1$$
If φ is 1-1 & onto, say $L_1 \cong L_2$

Key Example Adjoint Homomorphism

L a Lie Algebra. Define $ad: L \to gl(L)$ by

$$ad\,x\,(y) = [x,y]$$

- Linear since $[\,,\,]$ is bilinear
- $ad([x_1,x_2]) \stackrel{?}{=} [ad\,x_1, ad\,x_2]$

  check via Jacobi