Lecture 17 • go over homework

Dual Modules V, W L-modules, make $\mathrm{Hom}_F(V,W)$ an L-module via
$$(x\cdot\theta)(v) = x\cdot\theta(v) - \theta(x\cdot v)$$

Rmk When $W=F$ w/ trivial module structure then $(x\cdot\theta)(v) = -\theta(x\cdot v)$, $\theta\in V^*$

Let V have basis $\mathcal{B}=\{v_1, v_2, \dots, v_n\}$, dual basis $\check{\mathcal{B}}=\{\check{v}_1, \check{v}_2, \dots, \check{v}_n\}$ of V^*

$$\check{v}_i(v_j) = \delta_{ij}.$$

Observe Let $\theta = \sum c_i \check{v}_i \in V^*$ Then $c_i = \theta(v_i)$ so:

$$\theta = \sum_{i=1}^{n} \theta(v_i)\check{v}_i \qquad (*)$$

Let $x\in L$, $\varphi_V(x) = A$ matrix in terms of $\mathcal{B}$, so:

$$x\cdot v_j = \sum_{k=1}^{n} A_{kj} v_k. \qquad \text{What is } x\cdot\check{v}_j\text{ ?}$$

$$(x\cdot\check{v}_j)(v_k) = -\check{v}_j(x\cdot v_k) = -\check{v}_j\left(\sum_{s=1}^{n} A_{sk} v_s\right)$$

$$= \sum_{s=1} -A_{jk} \quad \text{so}$$

$$x\cdot\check{v}_j = \sum_{k=1}^{n} -A_{jk} v_k \quad (**)$$

Comparing $(*)$, $(**)$ we see matrix of $\varphi_{V^*}(x)$ in terms of $\check{\mathcal{B}}$ is $-A^{tr}$.

Cor If $\exists$ basis $\mathcal{B}$ so all $[\varphi(x)]_{\mathcal{B}}$ are skew symmetric, then $V\cong V^*$.

Rmk **HW** $\mathrm{Hom}_L(V,W) = \mathrm{Hom}_P(V,W)^L$ ← 0 weight space.

Review $L/\mathbb{C}$ Lie alg w/ Killing form κ.

Thm 1. L is solvable iff $L' \subseteq L^\perp$.
2. L is semisimple iff $L^\perp = 0$.

Cor 1 L is semisimple iff L is a direct sum of simple Lie algebras.

Cor 2 Every derivation of a semisimple Lie algebra is inner.

Long term Goal Classify all simple (hence semisimple) Lie algebras over $\mathbb{C}$
Tools 1. $ad: L \hookrightarrow gl(L)$
2. Results above, sl_2 results, abstract Jordan Dec.

Recall $V/\mathbb{C}$, $x \in gl(V)$ has a unique Jordan dec. $x = d + n$, d diagonalisable, n nilpotent, and $[d,n] = 0$. Further d, n are polynomials in x.

Rmk For L arbitrary, $\varphi: L \to gl(V)$ Jordan dec of $\varphi(x)$ isn't so useful, e.g. $L = \langle x \rangle$, $\varphi(x)$ is an arbitrary matrix.

Idea: L semisimple $/\mathbb{C}$, $ad: L \hookrightarrow gl(L)$, we can make this work.

Recall $T: V \to V$ linear, the generalized λ-eigenspace is

$$V_\lambda = \{ v \in V \mid (T - \lambda I)^m v = 0 \text{ for some } m \geq 0 \}$$

For $J_k(\lambda)$ this is entire space. So over $\mathbb{C}$, V_λ is just span of all basis vectors corr to Jordan blocks of e-value λ.

Prop Let $L/\mathbb{C}$ be a Lie alg. Let $\delta \in \mathrm{Der}\, L \subseteq gl(L)$. Suppose δ has Jordan Dec $\delta = \sigma + \nu$. Then σ, ν are both derivations

Proof Let $L_\lambda = \{x \in L \mid (\delta - \lambda\, \mathrm{Id})^m x = 0 \text{ some } m \geq 1\}$. Then

$$L = \bigoplus_{\substack{\lambda \text{ evalue} \\ \text{of } \delta}} L_\lambda, \text{ and note } L_\lambda \text{ is } \lambda\text{-eigenspace of } \sigma.$$

HW $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$ (use def of derivation!

By HW, for $x \in L_\lambda$, $y \in L_\mu$, $[x,y] \in L_{\lambda+\mu}$ so

$$\sigma([x,y]) = (\lambda+\mu)[x,y]$$

$$[\sigma(x), y] + [x, \sigma(y)] = [\lambda x, y] + [x, \mu y] = (\lambda+\mu)[x,y]$$

Thus σ is a derivation and so is $\eta = \delta - \sigma$. //

Thm Let L be a complex, semisimple Lie alg. Every $x \in L$ can be written uniquely as $x = d + n$ with $[d,n] = 0$, ad d diagonalizable, ad n nilpotent.

Moreover, if $[x,y] = 0$ then $[d,y] = [n,y] = 0$.

Def $x = d + n$ is abstract Jordan Decomposition.
If $n = 0$ say x is <u>semisimple element</u>.

Pf Write $\mathrm{ad}\, x = \sigma + \nu$ w/ $[\sigma, \nu] = 0$, σ diagonaliz, ν nilpotent. By the proposition, σ, ν are both derivations. By Cor 2 they are inner, so $\sigma = \mathrm{ad}\, d$, $\nu = \mathrm{ad}\, n$. So $\mathrm{ad}\, x = \mathrm{ad}(d+n)$, but ad is faithful so $x = d + n$. Now $\mathrm{ad}([d,n]) = [\mathrm{ad}\, d, \mathrm{ad}\, n] = 0$

so $[d,n] = 0$.

<u>Rmk</u> To get abs J.D of x, just "pull back" usual J.D of $\mathrm{ad}\, x$.

<u>Pf of Moreover</u> Suppose $[x,y] = \mathrm{ad}\, x(y) = 0$.

Write $n = c_0 \mathrm{Id} + c_1 \mathrm{ad}\, x + \cdots + c_s (\mathrm{ad}\, x)^s$. Then $n(y) = c_0 y$, but n is nilp so $c_0 = 0$ so $n(y) = [n,y] = 0$

But $[x,y] = [n,y] = 0 \Rightarrow [d,y] = 0$.

<u>Rmk</u> Suppose $x \in \overset{L^{ss}}{L} \subseteq \mathfrak{gl}(V)$. Then x has a Jord Dec as a member of $\mathfrak{gl}(V)$.
It also has an abstract J.D coming from $\mathrm{ad}\, x \in \mathfrak{gl}(L)$.
They are the same.

<u>Main Thm</u> Suppose L is semisimple, $\varphi : L \to \mathfrak{gl}(V)$ a representation.
Suppose $x \in L$ has abst JD $x = d + n$.

Then $\varphi(x) = \varphi(d) + \varphi(n)$ is the Jordan Dec of $\varphi(x)$.

<u>Pf.</u> $\varphi : L \to \varphi(L) \cong L/\ker \varphi$. Note $\varphi(L)$ is also semisimple

<u>Lemma</u> $L \twoheadrightarrow L'$ surjection, $x = d + n$ abs JD $\Rightarrow \varphi(x) = \varphi(d) + \varphi(n)$ abst J.D in L'!

Pf Use T/F from chpt 2, $\mathrm{ad}\, d$ diag $\Leftrightarrow \mathrm{ad}(\varphi(d))$ is, if φ ont.

But Rmk above says $\varphi(x) = \varphi(d) + \varphi(n)$ is also the ordinary Jordan Dec. //

** This thm will be applied repeatedly.