HW 8.2, 8.3, 8.5, 9.5, 9.6, 9.7

Lecture 15

Notation $GL(n,F)$ = general linear group = invertible $n\times n$ matrices w/ entries in F

Recall If $A \in gl(n,F)$ represents a linear map, and $P \in GL(n,F)$ then PAP^{-1} represents the same linear map in a different basis

Problem $GL(n,F)$ acts on $gl(n,F)$ via conjugation, $P \circ A = PAP^{-1}$
Understand the orbits? Come up with a system of orbit representatives

Rmk If $B = PAP^{-1}$ we know A,B have same char poly, e-values, dim(e-spaces), trace, rank, etc... Even if all these agree, not enough to conclude matrices are in same orbit.

Jordan Canonical Form

- If F is alg closed, there is a nice set of orbit reps, so two matrices are conjugate if & only if they have the same JCF. We work over $\mathbb{C}$.

Def Let $\lambda \in \mathbb{C}$. A $t\times t$ Jordan block w/ e-value λ is

$$J_t(\lambda) = \begin{array}{cc} & t \\ t & \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & 0 \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \; 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \; \lambda \end{pmatrix} \end{array}$$

Observe

1. $\operatorname{charpoly}(J_t(\lambda)) = (x-\lambda)^t = \operatorname{minpoly}(J_t(\lambda))$

2. The dimension of the λ-eigenspace of $J_t(\lambda)$ is 1.
(note $\operatorname{Rank}(J_t(\lambda) - \lambda\, Id) = t-1$)

3. The dimension of the generalized λ-eigenspace of $J_t(\lambda) = t$

Rmk This is $\{v \mid (T-\lambda I)^m v = 0 \text{ for some } m\}$

Thm Every $n\times n$ matrix over $\mathbb{C}$ is conjugate to a matrix which is block diagonal with Jordan blocks on the diagonal. Two matrices are conjugate iff they have the same Jordan Canonical Form (JCF), up to possibly reordering the blocks.

Ex
$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} J_3(3) & & \\ & J_2(2) & \\ & & J_1(2) \end{pmatrix}$$

Rmk

1. The dimension of λ-eigenspace is the # of blocks $J_i(\lambda)$.
2. The dimension of generalized λ-eigenspace is the sum of the sizes of blocks $J_{t_i}(\lambda)$
3. The JCF determines charpoly, minpoly, but (if $n\geq 4$) not vice versa.
4. A is diagonalizable iff all Jordan blocks are 1×1
5. A matrix is nilpotent $\leftrightarrow$ all e-values are zero. Thus JCFs of nilpotent matrices correspond to partitions of n.

$n=4$ $\quad \begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$ part (4) $\quad \begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$ $(3,1)$

$\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$ $(2,2)$ $\quad \begin{pmatrix} 0&1&&\\0&0&&\\&&0&\\&&&0 \end{pmatrix}$ $(2,1,1)$ $\quad (0)$ $(1,1,1,1)$

COR Any linear transformation x of $V/\mathbb{C}$ has a Jordan decomposition $x=d+n$ where d is diagonalizable, n is nilpotent, and d & n commute. This decomposition is unique.

Pf Let PAP^{-1} be in JCF. Clearly holds for $PAP^{-1}=\tilde{d}+\tilde{n}$, $\tilde{d}$ diagonal.
Then $A=P^{-1}(\tilde{d}+\tilde{n})P=P^{-1}\tilde{d}P+P^{-1}\tilde{n}P:=d+n$ //

<u>Technical Lemma</u> Let $x = d + n$ as above.

(a) $\exists$ a polynomial $p(t) \in \mathbb{C}[t]$ so $d = p(x)$.

(b) Suppose $PxP^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, so $x = P^{-1}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}P$. Let $P^{-1}\begin{pmatrix} \bar\lambda_1 & & \\ & \ddots & \\ 0 & & \bar\lambda_n \end{pmatrix}P$ be $\bar d + n$. Then $\exists\, q(t) \in \mathbb{C}[t]$ so $q(x) = \bar d$.

<u>Killing Form</u>

<u>Problem</u> Given L, how to determine if L is semisimple (= no solvable ideals) or if L is solvable, or neither?

<u>Def</u> Let L be a complex Lie alg. The <u>Killing Form</u> is the symmetric, bilinear form defined by

$$\kappa(x,y) = tr(\mathrm{ad}\,x \circ \mathrm{ad}\,y).$$

<u>Rmk</u> The identity $tr([a,b]c) = tr(a[b,c])$ implies

$$\kappa([x,y],z) = \kappa(x,[y,z]),$$ sort of associativity.

<u>Thm</u>

1. (Cartan's First Criterion) $L/\mathbb{C}$ is solvable $\leftrightarrow \kappa(x,y) = 0 \ \forall x \in L, y \in L'$.

2. (Cartan's Second Criterion) $L/\mathbb{C}$ is semisimple $\leftrightarrow \kappa$ is nondegenerate.

<u>Proof of First Criterion</u>

$\Rightarrow$ L is solvable iff $\mathrm{ad}\,L$ is solvable. Choose basis of L so all matrices are upper triangular. Then since $y \in L'$, matrix $\mathrm{ad}\,y$ is strictly upper Δ. And $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\ & 0 \end{pmatrix}$ so $\mathrm{ad}\,x \circ \mathrm{ad}\,y$ is strictly upper Δ.

$\Leftarrow$ Since L is solvable iff $\mathrm{ad}\,L$ is solvable, WLOG assume we are working in matrix alg.

ETS V/ϕ

Prop 8.3 Let $L \leq gl(V)$, suppose $tr(xy) = 0$ $\forall x \in L, y \in L'$. Then L is solvable.

Proof We show instead that L' is nilpotent by showing every $w \in L'$ is a nilpotent linear map. So let

$$w = d + n \in L' \quad \text{Need } d = 0 \text{ (all e-values of } w \text{ are } 0\text{)}.$$

Lemma 1 Suppose $w = d + n$ is Jordan Dec. Then so is $\operatorname{ad} w = \operatorname{ad} d + \operatorname{ad} n$.

Pf n nilpotent $\Rightarrow$ $\operatorname{ad} n$ nilp done in class

d diagonalizable $\Rightarrow$ $\operatorname{ad} d$ diagonalizable (HW: evalues $\{\lambda_i - \lambda_j\}$) //

Lemma 2 Let $x = d + n \in L$, $y \in L$. Then $[\bar{d}, y] \in L'$ (Note $\bar{d}$ may not be in L)

Proof By techical Lemma, $\operatorname{ad} \bar{d}$ is a polynomial in $\operatorname{ad} x$ (Since $\operatorname{ad} x = \operatorname{ad} d + \operatorname{ad} n$)

Thus $[\bar{d}, y] \in L'$ since $\operatorname{ad} x$ maps $L \to L'$. //

Back to $w = d + n \in L'$. Choose a basis so d is diagonal,

$$d = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}. \quad \text{Note } tr(\bar{d} w) = \sum \lambda_i \bar{\lambda}_i$$

so ETS $tr(\bar{d} w) = 0$. But w is a lin combo of brackets $[y, z]$ and

$$tr(\bar{d}, [y, z]) = tr([\bar{d}, y] z)$$

$$= tr(z [\bar{d}, y]) = 0 \text{ since } [\bar{d}, y] \in L'$$

//