Lecture 13

<u>Review</u> L a Lie algebra, V an L-module means:

- bilinear map $L \times V \to V$, $(x, \vec{v}) \to x \cdot \vec{v}$ such that
- $[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v \quad \forall x, y \in L, v \in V$

Corresponding <u>representation</u> $\varphi : L \to gl(V)$, $\varphi(x)(v) := x \cdot v$, is a Lie algebra homomorphism.

<u>Def</u> A <u>submodule</u> is an L-invariant subspace $U \subseteq V$.

<u>Def</u> V is <u>simple</u> (irreducible) if only submodules are $\{0\}$ V.

$$\text{simple} \longleftrightarrow \nexists \text{ basis of } V \text{ so } \varphi(L) \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

<u>EX</u> Submodule gen by collection of vectors in V.

<u>Def</u> Quotient module V/U, $x \cdot (v + U) := xv + U$

<u>Def</u> V is <u>indecomposable</u> if $\nexists$ submodules U, W with $V = U \oplus W$.

$$\text{indecomposable} \longleftrightarrow \nexists \text{ basis of } V \text{ so } \varphi(L) \subseteq \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

so irreducible $\Longrightarrow$ indecomposable.

<u>Rmk</u> Exercise: Let U, W be subspaces w/ $U \cap W = \{0\}$ and $V = \langle U, W \rangle$. Then $V \cong U \oplus W$ as vector spaces. If both are submodules then $V \cong U \oplus W$ as modules.

Given $U \subset V$ a submodule, there always is a <u>subspace</u> so $V = U \oplus W$ as vector spaces, but not always a submodule.

Homomorphisms

<u>Def</u> Let L be a Lie algebra, V, W L-modules. An <u>L-module homomorphism</u> is a linear map $T: V \to W$ such that

$$T(x \cdot v) = x \cdot T(v) \quad \forall x \in L,\ v \in V.$$

$$\begin{tikzcd} v \arrow[r, "T"] \arrow[d, "x"'] & T(v) \arrow[d, "x"] \\ x\cdot v \arrow[r, "T"] & T(x\cdot v) \end{tikzcd}$$

commutes

<u>Rmk</u> $\varphi_V: L \to gl(V)$, $\varphi_W: L \to gl(W)$ then

$$T \circ \varphi_V(x)(v) = \varphi_W(x) \circ T(v), \quad \text{i.e. } T \circ \varphi_V = \varphi_W \circ T.$$

sometimes say T <u>intertwines</u>.

<u>Exercise</u> Let $T: V \to W$ be a module homom. Then $\operatorname{Image}(T)$ and $\operatorname{Ker} T$ are both submodules (of W & V respectively)

Isomorphism Thms

1. Let $T: V \to W$ be L-module homomorphism. Then $V/_{\operatorname{Ker} T} \cong \operatorname{Im}(T)$ as L-modules
2. Let U, W be submodules of V. So are $U+W$ and $U \cap W$ and
$$U+W/_W \cong U/_{U \cap W} \text{ as } L\text{-modules}$$
3. $U \leq W \leq V$ then W/U is a submodule of V/U and $V/U\big/_{W/U} \cong V/W$.
4. Submodules of V/U $\leftrightarrow$ submodules of V containing U.

<u>Ex</u> L be 1-dim abelian, $\langle x\rangle$. Then $x \to f \in gl(V)$ is a representation for any f.

When is this $\cong$ to $x \to g \in gl(W)$?

Need $\cong T$ s.t. $Tf = gT$, i.e. $TfT^{-1} = g$, so same matrix, diff't basis

<u>Generalize</u> $\varphi_1 : L \to gl(n,F)$, $\varphi_2 : L \to gl(n,F)$ are $\cong$ iff $\exists$ single change of basis matrix sending $\varphi_1(x)$ to $\varphi_2(x)$ $\forall x \in L$.

7.9 Ex $L = \langle x, y \mid [x,y] = x\rangle$ over $\mathbb{C}$ Define

$$\varphi(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varphi(y) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

<u>Note</u> $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x$

so φ is a representation

<u>Claim</u> φ is $\cong$ to the adjoint rep ad: $x \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y \to \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ Basis $e_1, e_2 \rightsquigarrow \{e_1, e_1 + e_2\}$

$T: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

* $\mathrm{Hom}_L(V,W)$ is a vector space

* $\mathrm{Hom}_L(V,V) = \mathrm{End}_L(V)$ is an F-alg.

<u>Schur's Lemma</u> Let L be a Lie algebra $/\mathbb{C}$, S a simple L-module, $\theta : S \to S$ a module hom. Then $\theta = \lambda \cdot Id$

<u>Proof</u> Let $\theta: S \to S$. Since $/\mathbb{C}$, θ has an eigenvalue λ. Consider $\theta - \lambda Id$, also a module homomorphism. Then

$\mathrm{Ker}(\theta - \lambda Id) \neq 0$, but it's a submodule,

so $\mathrm{Ker}(\theta - \lambda Id) = S$ i.e. $\theta = \lambda Id$ //

Schur's Continued

- More generally if S is simple L-module, F arbitrary, then
$$\mathrm{End}_L(S) \text{ is a } \underline{\text{division algebra}}.$$

Important Corollary

Suppose $L/\mathbb{C}$ and S is irreducible. Let $z \in Z(L)$. Then z acts by scalar multiplication on S, i.e. $\exists \lambda \in \mathbb{C}$ so $z \cdot v = \lambda v \ \forall v \in S$

Proof Claim: Fix z. Then $v \mapsto z \cdot v$ is an L-module homom.

Proof Clearly linear. Let $L_z(v) = z \cdot v$

$$\begin{aligned} L_z(x \cdot v) &= z \cdot x \cdot v \\ &= x \cdot z \cdot v + [z, x] \cdot v \\ &= x \cdot L_z(v) + 0 \quad \text{since } z \in Z(L) \end{aligned}$$

Now apply Schur.

COR L abelian $/\mathbb{C} \Rightarrow$ all simple modules 1-dim (True for solvable)