Lecture 12

7.3, 7.4, 7.8, 7.10, 7.11, 7.12

<u>Representation Theory</u> L a Lie alg /F, V a vector space /F. How can L act on V?

Want: Map $L \times V \to V$ denoted $(x,v) \to x \cdot v$ such that:

$L \times V \to V$ Bilinear

1. x acts linearly: $x \cdot (\lambda v_1 + v_2) = \lambda x \cdot v_1 + x \cdot v_2 \quad \forall x \in L, v_i \in V, \lambda \in F$
2. Linear in L: $(\lambda x + y) \cdot v = \lambda x \cdot v + y \cdot v \quad \forall x, y \in L, v \in V, \lambda \in F$
3. $[x,y] \cdot \vec{v} = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \forall x, y \in L, v \in V$

<u>Def</u> V as above is called an <u>L-module</u>.

<u>Ex</u> Suppose $L \subseteq gl(V)$. Then V is an L-module via $x \cdot v = x(v)$,

<u>Rmk</u> Suppose V is an L-module. Then $x \cdot \_ : V \to V$ is a linear map

<u>Exercise</u> Let V be an L-module. Define $\varphi : L \to gl(V)$ by

$$\varphi(x)(v) = x \cdot v \quad (\text{i.e. } \varphi(x) = x \cdot \_)$$

Then φ is a Lie alg. homom.

"1" ensures $\varphi(x) \in gl(V)$, i.e. that $\varphi(x)$ is linear.
"2" ensures φ is a linear map
"3" ensures φ is a Lie alg. map:

$$\varphi([x,y])(v) = [x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$$
$$= x \cdot \varphi(y)(v) - y \cdot \varphi(x)(v) = \varphi(x)\varphi(y)(v) - \varphi(y)\varphi(x)(v)$$
$$= [\varphi(x), \varphi(y)](v)$$

Def A <u>representation</u> of L is a Lie alg. homomorphism $\varphi : L \to gl(V)$.

(**) 1. Given $\varphi : L \to gl(V)$, V is an L-module via $x \cdot v := \varphi(x)(v)$

2. Given an L-module V, get a representation by $\varphi(x)(v) = x \cdot v$.

Two equivalent ways of looking at the same thing.
* dimension of φ = dim V

Rmk Given a <u>matrix rep</u> $\varphi : L \to gl(n, F)$, the corresponding module is just column vectors F^n.

Ex Adjoint representation $ad : L \to gl(L)$. "L is an L-module via ad."

Next: After defining a new object (L-modules), next define submodules, quotient modules, module homomorphisms.

Def Let V be an L-module. A subspace $U \subseteq V$ which is L-invariant is called a <u>submodule</u>. (L-invariant: $x \cdot \vec{u} \in U$ $\forall x \in L, u \in U$)

Rmk Picking a basis of U and extending, we see the corresponding representation $\varphi : L \to gl(n, F)$ has:

$$\varphi(L) \subseteq \begin{array}{c} \\ U \end{array}\!\!\begin{array}{c} U \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}$$

Examples 0. $\{0\}$ and V are always submodules

1. Consider L as an L-module via ad. Then submodules of L are precisely ideals of L.

2. Consider $b(n, F)$ w/ natural module F^n. Lots of submodules!

3. <u>Invariance Lemma</u> char $F=0$, $A \subseteq L \subseteq gl(V)$, A an ideal. Let V_λ be an A-weight space. Then V_λ is an L-submodule of V.

4. Suppose $F=\mathbb{C}$ and L is solvable. Then any L-module has a one-dimensional submodule, by Lie's Thm.

5. Let U, V be L-modules. Check $U \oplus V$ is an L-module via $x\cdot(u,v)=(xu,xv)$ with submodules $\{(u,0)\}$ and $\{(0,v)\}$.

6. submodule generated by $\{v_1, \dots, v_s\} \subseteq V$.

<u>Def</u> An L-module is irreducible (or simple) if it has no nontrivial submodules

<u>Examples</u>

1. Any 1-dimensional module is simple
2. Suppose L is simple (no nontrivial ideals). Then L is a simple L-module via ad, and vice versa.
3. Lie's Thm says, over $\mathbb{C}$, that any simple module for a solvable Lie algebra is one-dimensional!
4. The natural module $\mathbb{C}^2$ for $sl(2,\mathbb{C})$ is simple.
5. V is simple if & only if it is generated by any nonzero $v \in V$.

Quotient Modules

<u>Def</u> Let V be an L-module with submodule U. Recall $V/U = \{v+U\}$ with $v_1 + U = v_2 + U \leftrightarrow v_1 - v_2 \in U$

Define an action of L on V/U by $x \cdot (v+U) := x \cdot v + U$.

<u>Claim</u> V/U is an L-module, called the <u>quotient module</u>.

Well-Defined: If $v_1 + U = v_2 + U \Rightarrow v_1 - v_2 \in U \Rightarrow x \cdot (v_1 - v_2) = xv_1 - xv_2 \in U$
$\Rightarrow x \cdot v_1 + U = x \cdot v_2 + U$ ✓

Other axioms are immediate.

<u>Exercise</u> Suppose $U \leq V$. Pick a basis and extend.

$$\varphi(x) \in \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Let $$\varphi(x) = \begin{pmatrix} \varphi_U(x) & * \\ 0 & \varphi_{V/U}(x) \end{pmatrix}$$

Check $\varphi_U(x)$ is rep for U, $\varphi_{V/U}(x)$ is rep of V/U.

<u>Rmk</u> V/U is not, in general, a submodule

<u>Def</u> V is <u>indecomposable</u> if $\nexists$ nonzero submodules U, W with $V = U \oplus W$.

* Irreducible $\Longrightarrow$ Indecomposable
$\not\Longleftarrow$
EX $b(Q,F)$