Read Chpt 7

Lecture N

Review Engel's Thm Suppose $L \leq gl(V)$ a Lie subalgebra and suppose every $x \in L$ is nilpotent. Then $\exists$ a basis $\mathcal{B}$ so that $[x]_{\mathcal{B}} \in n(n,F)$ $\forall x \in L$. In particular, L is nilpotent.

Cor A Lie algebra is nilpotent if & only if every element of L is ad-nilpotent.

Next: Analyze Solvable subalgebras of $gl(V)$, note Engel over any field.

Lie's Theorem Suppose V is n-dim $\mathbb{C}$ vector space, $L \leq gl(V)$ is a solvable subalgebra. ~~of~~ Then $\exists$ a basis $\mathcal{B}$ of V so that $[x]_{\mathcal{B}} \in b(n,\mathbb{C})$ $\forall x \in L$.

Proof For Engel we needed a $v \in V$ so $xv = 0$ $\forall x \in L$. Here we first need a $v \in V$ that is a simultaneous eigenvector for all $x \in L$.

Easy Case $L = \langle x \rangle$ one-dimensional $F = \mathbb{C}$ so $\exists$ an eigenvector $v \in V$. Let $U = \langle v \rangle$, use induction on $\bar{x} : V/U \to V/U$ as before.

Prop Let V as above, $L \leq gl(V)$ solvable subalgebra. Then $\exists$ $0 \neq v \in V$ a simultaneous eigenvector for L.

Proof Induction on dim L, base case above

Since L is solvable, $L' \subsetneq L$. Choose a codimension one subspace $A \geq L'$, so

$$L = A \oplus \langle z \rangle \text{ as vector spaces, some } z \neq 0.$$

Recall L/L' is abelian, so any subspace is an ideal, Thus A is an ideal.
$([x,a] \in L' \subseteq A)$

Thus A is a subalgebra of smaller dimension, so by induction $\exists$ a nonzero weight space for $\lambda : A \to \mathbb{C}$

$$0 \neq V_\lambda = \{ v \in V \mid \lambda(a)v = av \quad \forall a \in A \}$$

By invariance lemma, V_λ is L-invariant, so $z : V_\lambda \to V_\lambda$.
Over $\mathbb{C}$ so choose $0 \neq v \in V_\lambda$ with $zv = \mu v$.

Any $x \in L$ is of form $a + cz$, $c \in \mathbb{C}$ and $xv = (a+cz)v$
$= (\lambda(a) + c\mu)v$

so v is the common eigenvector. //

Rmk Lie's Thm holds over any alg closed field of characteristic 0.

Assume $F = \bar{F}$ char 0

Cor 1 Suppose L is solvable. Then $\exists$ a chain of ideals
$$0 = L_0 \subset L_1 \subset \dots \subset L_n = L \text{ with } \dim L_i = i$$

Proof Consider $\mathrm{ad} : L \to \mathfrak{gl}(L)$. Then $\mathrm{ad}(L)$ is solvable. By Lie, $\exists$ a basis $\{v_1, v_2, \dots, v_n\}$ of L so every $\mathrm{ad}\, x$ is upper Δ

Let $L_i = \mathrm{span}\{v_1, v_2, \dots, v_i\}$ //

Cor 2 Suppose L is solvable. Then $L^{(1)} = [L, L]$ is nilpotent.

Proof By Cor 1 $\exists$ a basis $\{v_1, v_2, \dots, v_n\}$ of L so each $\mathrm{ad}\, x \in \mathfrak{b}(n, F)$. Thus matrices of $\mathrm{ad}([L,L]) = [\mathrm{ad}\, L, \mathrm{ad}\, L]$ lie in $\mathfrak{n}(n, F)$. Thus every elt of $[L,L]$ is ad-nilpotent, so by Engel, $[L,L]$ is nilpotent.

Representation Theory

Problem Given a Lie algebra L, in what ways can we express L (or maybe L/I) as a subalgebra of a matrix algebra?

Def Let L be a Lie algebra $/F$. A representation of L is a Lie algebra homomorphism:

$$\varphi : L \to gl(V)$$

Rmk $L/\ker\varphi \cong \varphi(L) \leq gl(L)$

Def φ is faithful if $\ker\varphi = 0$. In this case $L \cong \varphi(L) \leq gl(V)$.

Rmk If $\varphi : L \to gl(n,F)$, say φ is a matrix representation.

Examples

1. Most important: adjoint representation $ad : L \to gl(L)$
This is faithful if & only if $Z(L) = 0$.

2. Suppose $L \leq gl(V)$. Inclusion map $L \hookrightarrow gl(V)$ called the natural representation

Ex $sl_2(\mathbb{C})$ $e \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2-dimensional nat. rep.

Ex $ad : sl_2(\mathbb{C}) \to gl(sl_2(\mathbb{C}))$

$[e,e]=0,\ [e,f]=0e+0f+h \quad [e,h]=-2e+0f+0h$

$$e \to \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$[f,e]=0e+0f-h \quad [f,f]=0 \quad [f,h]=2f$

$$f \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

$[h,e]=2e \quad [h,f]=-2f \quad [h,h]=0 \qquad h \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3 dimensional representation

Ex Trivial representation $\quad \varphi(x)=(0) \quad \forall x \in L$.

Ex $\mathbb{R}^3_\wedge = \langle i,j,k \mid [i,j]=k,\ [j,k]=i,\ [k,i]=j \rangle$

adjoint rep $\quad i \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad j \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$k \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex Exc 1.15, $\mathbb{R}^3_\wedge \cong \left\{ \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \right\}$

Modules

Suppose $\varphi: L \to gl(V)$ a repr. For $x \in L$, $v \in V$ let x act on V by

$$x \cdot v = \varphi(x)(v).$$

$$* \quad (\lambda x + \mu y) \cdot v = \lambda x \cdot v + \mu y \cdot v$$
$$x(\lambda v + \mu w) = \lambda x \cdot v + \mu x \cdot w$$

$$\begin{aligned} [x,y] \cdot v = \varphi([x,y])(v) &= [\varphi(x), \varphi(y)](v) \\ &= \varphi(x)\varphi(y)v - \varphi(y)\varphi(x)v \\ &= x \cdot (y \cdot v) - y \cdot x \cdot v \end{aligned}$$

Say V is an L-module

Think Linear action of L on V such that
$[x,y] \cdot v = x \cdot y(v) - y \cdot x(v)$