Name: SOLUTIONS

Math 461/561 Midterm Exam - October 21, 2010

1. **(20 points)** Complete the following:

a. Let L_1 and L_2 be Lie algebras over F. A linear map $\phi : L_1 \to L_2$ is a *Lie algebra homomorphism* if $\cdots$

$$\phi([x,y]) = [\phi(x), \phi(y)] \quad \forall x,y \in L_1.$$

b. Let L be a Lie algebra and A a subalgebra. The *normalizer* of A in L, denoted $N_L(A)$ is $\cdots$

$$N_L(A) = \{x \in L \mid [x,a] \in A \quad \forall a \in A\}$$

c. Let A be a Lie subalgebra of $gl(V)$. A *weight* for A is $\cdots$

a linear functional $\lambda \in A^*$ such that the weight space $V_\lambda = \{v \in V \mid \lambda(a)v = a(v) \quad \forall a \in A\}$

is nonzero.

d. A Lie algebra L is *solvable* if $\cdots$ $L^{(k)} = 0$ for some k,

where $L^{(1)} = [L,L]$

$L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$

2. **(20 points)** True or false. If false, give a counterexample.

_T_ a. Every irreducible representation of a Lie algebra is indecomposable.

_F_ b. Suppose L is a nilpotent Lie subalgebra of $gl(V)$. Then every element of L is nilpotent.

$$L = \langle Id \rangle \subseteq gl(V)$$

_T_ c. The kernel of a Lie algebra homomorphism is an ideal.

_F_ d. Every solvable Lie algebra is nilpotent.

$$L = \langle x, y \mid [x, y] = x \rangle$$

_T_ e. Every irreducible representation of a solvable Lie algebra over $\mathbb{C}$ is one-dimensional.

_F_ f. The adjoint representation of $sl(2, F)$ is faithful for any field.

$\ker ad = Z(L)$

and $Z(sl(2,F)) \neq 0$ if char $F = 2$

3. **(20 points)** Let L be a Lie algebra over a field F. Define what a representation of L is and what an L-module is, and explain how one goes back and forth between representations of L and L-modules.

A representation is a Lie algebra homomorphism

$$\varphi : L \to \mathfrak{gl}(V) \quad \text{for some vector space } V/F.$$

An L-module is an F-vectorspace V together with a bilinear map

$$L \times V \to V \qquad \text{such that}$$
$$(x, v) \to x \cdot v$$

$$[x,y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v \quad \forall x, y \in L, \ \forall v \in V.$$

Given $\varphi : L \to \mathfrak{gl}(V)$, V is an L-module via

$$x \cdot v := \varphi(x)(v)$$

Given an L-module V, the corresponding representation is

$$\varphi : L \to \mathfrak{gl}(V)$$

$$\varphi(x)(v) =: x \cdot v$$

4. **(20 points)** Recall that a derivation of a Lie algebra L is a linear map $D : L \to L$ (i.e. $D \in gl(L)$) such that:

$$D([x, y]) = [Dx, y] + [x, Dy] \;\; \forall x, y \in L.$$

i. For $x \in L$ prove $\operatorname{ad} x$ is a derivation. Derivations of this form are called *inner*.

ii. For derivations D_1, D_2 of L, prove $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation of L, i.e. the set of derivations, denoted Der$L$, is a Lie subalgebra of $gl(L)$.

iii. Prove the set of inner derivations is an ideal of Der L. You may assume it is a subspace.

1. $\operatorname{ad} x([a,b]) = [x,[a,b]] = -[a,[b,x]] - [b,[x,a]]$ by Jacobi Id.

$= [a,[x,b]] + [[x,a],b]$ by skew sym.

$= [a, \operatorname{ad}x(b)] + [\operatorname{ad}x(a), b]$ //

2. $[D_1, D_2][x,y] = D_1 \circ D_2([x,y]) - D_2 \circ D_1([x,y])$

$= D_1([D_2x,y] + [x,D_2y]) - D_2([D_1x,y] + [x,D_1y])$

$= [D_1D_2x,y] + \cancel{[D_2x,D_1y]} + \cancel{[D_1x,D_2y]} + [x,D_1D_2y]$

$- [D_2D_1x,y] - \cancel{[D_1x,D_2y]} - \cancel{[D_2x,D_1y]} - [x,D_2D_1y]$

since D_1, D_2 are derivations

$= [D_1D_2x,y] + [x,D_1D_2y] - [D_2D_1x,y] - [x,D_2D_1y]$

$= [(D_1D_2 - D_2D_1)x, y] + [x,(D_1D_2 - D_2D_1)y]$

$= [[D_1,D_2]x,y] + [x,[D_1,D_2]y]$ //

5. **(20 points)** Let $L = sl(2, \mathbb{C})$ and let V be a finite-dimensional ~~F~~ L-module. Consider the subalgebra spanned by $\{e, h\}$, denoted B. (You should not assume the results proved in class on Tuesday from Chapter 8!)

i. Show that B is a solvable subalgebra of L.

ii. Use Lie's theorem to prove there exists $0 \neq w \in V$ such that w is an eigenvector for h and $e \cdot w = 0$. Hint: V is an L-module so by restriction it is also a B-module with corresponding representation $\phi_V : B \to gl(V)$.

i. $B = \langle e, h \mid [h,e] = 2e \rangle$ so $B' = \langle e \rangle$, $B^{(2)} = 0$ so B is solvable.

ii. Since V is a B-module we have:

$\phi_V : B \to gl(V)$ $\quad$ B is solvable

so $\phi_V(B)$ is a solvable subalgebra of $gl(V)$. By Lie's Thm (since $F = \mathbb{C}$) there is a basis of V so $\phi_V(B)$ is all upper triangular matrices, call it $B = \{v_1, v_2, .., v_n\}$.

But $e = [h, \frac{1}{2}e]$ so $\phi_V(e) = \frac{1}{2}(\phi_V(h)\phi_V(e) - \phi_V(e)\phi_V(h))$.

Thus $\phi_V(e)$ is a bracket of 2 upper Δ matrices, so is strictly upper Δ. Thus in B:

$$\phi(h) = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \quad \phi(e) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

Thus $W = V_1$ is an e-vector of h and $e.w = 0$. //

iii. Let $D \in \mathrm{Der} L$, $\mathrm{ad}\, x \in I\mathrm{Der} L$. Then for $y \in L$:

$$[D, \mathrm{ad}\, x]\, y = D \circ \mathrm{ad}\, x\, y - \mathrm{ad}\, x \circ D y$$
$$= D([x,y]) - [x, Dy]$$
$$= [x, Dy] + [Dx, y] - [x, Dy]$$
$$= [Dx, y]$$

Thus $[D, \mathrm{ad}\, x] = \mathrm{ad}\,(Dx)$ is inner.

So $I\mathrm{Der} L$ is an ideal in $\mathrm{Der} L$.