9. It is not a homomorphism because it is not well defined. For example in $Z_{12}$, $3 = 15$. However $\phi(3) = 9$ and $\phi(15) = 45$ but $9$ and $45$ are not congruent mod $10$!

15. See back. Recall that all the elements in a single coset of the kernel map to the same element.

16. If there was a homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$ it would have to be an isomorphism, it is assumed onto but both groups have 16 elements so it would also be one-to-one. However these groups are clearly not isomorphic, the one on the left has elements of order 8 why the one on the right has maximum order of an element being 4.

21. By the 1st $\cong$ theorem $Z_{30}/\ker \phi$ is isomorphic to the image, which has 5 elements. Thus the kernel has 6 elements. $Z_{30}$ has a unique subgroup of order 6, namely $\{0, 5, 10, 15, 20, 25\}$.

30. This follow from theorem 9. The kernel has order 5. Given a normal subgroup $H$ of $Z_6 \oplus Z_2$ or order $a$, then $\phi^{-1}(H)$ is a normal subgroup of $G$ or order $5|H|$. However $Z_6 \oplus Z_2$ is abelian so every subgroup is normal. It has subgropus of order 1, 2, 3, 4, 6 and 12 which correspond to normal subgroups of $G$ of orders 5, 10, 15, 20, 30, 60.

47. See back. It can’t be finite since the 1st isomorphism theorem would guarantee every prime divides its order.

54. Define a map $\phi : G \to G/H \times G/K$ by $\phi(g) = (gH, gK)$. Check that this is a homomorphism. Since the identity in $G/H \times G/K$ is $(H, K)$, we see that $g$ is in the kernel of $\phi$ exactly when $gH = H$ and $gK = K$, i.e. when $g \in H \cap K$. But $H \cap K = \{e\}$. Thus the first isomorphism theorem says $G \cong \phi(G)$ and $\phi(G)$ is a subgroup of $G/H \times G/K$.

8. It is $\mathbb{Z}/k\mathbb{Z}$.

9. See back. To get the center just pick a matrix $Z \in Z(H)$. Write down the equation for $Z$ to commute with an arbitrary element of $H$ and see that this forces $a = c = 0$. 
26. Let $z = a + bi$ be a complex number. Recall that the norm $|z| = \sqrt{a^2 + b^2}$ so $T$ is just the set of $z$ of norm 1. Clearly $R^+$ and $T$ are subgroups under multiplication. Note that their intersection is just $1 = 1 + 0i$, and they are normal because $C^*$ is abelian. Finally for any $z \in C$ we have:

$$z = |z| \frac{z}{|z|} \in R^+T$$

so $C^* = R^+T$. This proves $C^*$ is the internal direct product of $R^+$ and $T$.

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2. The unity in this ring is 6.

6. a. In $Z_6$, $3^2 = 3$.
   b. In $Z_6$, $3 \cdot 2 = 0$ but neither 3 nor 2 is 0.
   c. In $Z_{12}$, $3 \cdot 4 = 3 \cdot 8$ but 4 $\neq$ 8.

None of our $n$’s are prime. Indeed the ring $Z_p$ is a field, these properties all do hold in a field.

20. The elements in $M_2(Z)$ with multiplicative inverses are those with determinant $\pm 1$.

22. If $a$ and $b$ are units then $(ab)(b^{-1}a^{-1}) = 1$ so $ab$ is also a unit. Also since $(a^{-1})^{-1} = a$, we see that $a^{-1}$ is a unit. Multiplication is associative and 1 is a unit, so we have checked all the axioms to see $U(R)$ is a group under multiplication.

28. $4 = 2 \cdot 5$ in $Z_6$ so $4 \mid 2$. Also $7 = 3 \cdot 5$ in $Z_8$, and $12 = 9 \cdot 3$ in $Z_{15}$.

50. Let $R$ Boolean. First notice that $(x + x) = (x + x)^2 = x^2 + x^2 + x^2 + x^2 = x + x + x + x$. Subtracting gives $x + x = 0$, i.e. $x = -x$ for any $x$ in a Boolean Ring. Now let $x, y \in R$. Then:

$$a + b = (a + b)^2$$
$$= a^2 + ab + ba + b^2$$
$$= a + ab + ba + b$$

Subtracting gives $ab = -ba$ but we know $-ba = ba$ so $ab = ba$. 