9. If \( p \mid H \), so is any multiple of \( p \). So \( pG \to p \in H \) and \( p^a \in H \), which is \( e \).

27. \( axb=c \) \( \Rightarrow \) \( b^{-1}ca \in H \)

Thus \( x^{-1}x a = c \) \( \Rightarrow \) \( x^{-1} = aca^{-1} \)

29. a. Suppose \( x^3 = e \), and assume \( x \neq e \). If \( x^2 = e \) then \( x^2 = e \) so \( x^3 = e = e = e \).

This \( x^3 = e \) means \( x^{-1} \neq x \). Hence any \( x \) with \( x^3 = e \) means \((x^{-1})^3 = e \) and \( x^{-1} \neq x \).

Thus elements which have \( x^3 = e \) are \( x = e \) and \( p \) pairs \( x \neq x^{-1} \). This there is an odd \#.

b. Similar to a, if \( x^2 = e \) then \((x^{-1})^2 = e \) and \( x \neq x^{-1} \) so there is an even \# of such elements.
34. Matrix Multi is associative.

The identity \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is in \( H \).

It is closed under multiplication.

Finally observe that \( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \) \( \forall \) \( h \in H \).

Thus \( H \) is a group.

37. \( G = \{ \begin{pmatrix} a & b \\ a & a \end{pmatrix} | a \in \mathbb{R}, a \neq 0 \} \)

1. Notice \( \begin{pmatrix} a & b \\ a & a \end{pmatrix} \begin{pmatrix} c & d \\ c & c \end{pmatrix} = \begin{pmatrix} ac + ad & bc + bd \\ ac + ad & ac + ad \end{pmatrix} \in G. \)

So \( G \) is closed under matrix multiplication.

2. \( \begin{pmatrix} a & b \\ a & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a \end{pmatrix} \)

\( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} a \end{pmatrix} \) so \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) is the identity.

3. \( \begin{pmatrix} a & b \\ a & a \end{pmatrix} \begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \)

\( \begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix} \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \)

So the inverse of \( \begin{pmatrix} a & b \\ a & a \end{pmatrix} \) is \( \begin{pmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{pmatrix} \) in this group.
#32

This is a Forest.
also \( r^{-1} \) so

\[
fr = r^{-1} \Rightarrow \begin{cases} 
fr = r^{-1}f \\
fr^{-1} = rf
\end{cases}
\]

a. \( D_4 \)

\[fr^2fr^5 = frfrfr = r^{-1}frfr = r^{-1}r^{-1}frfr = r^{-1}r^{-1}frfr = r^{-1} = r^3\]

b. \( D_5 \)

\[r^3fr^{-3}fr^{-3} = rrfr^{-1}fr^{-1} = rtr^{-1}fr^{-1} = rtfr^{-1}fr^{-1} = r^4 = r\]

c. \( D_6 \)

\[fr^5fr^{-3} = fr^{-1}fr^{-1}fr^{-1}x = rtr^{-1}fr^{-1}fr^{-1}x = r^5f = r^5\]
1. \[ \begin{align*}
11 & = 11 \quad 101 = 1 \\
111 & = 181 = 171 = 1111 = 12 \\
121 & = 110 = 4 \quad 6 \\
131 & = 191 = 4 \\
141 & = 181 = 3 \\
161 & = 2
\end{align*} \]

\[ \begin{align*}
11(10) & = \{1, 3, 7, 9, 3\} \\
11(100) & = 4 \\
11 & = 1 \\
131 & = 4 \\
171 & = 4 \\
181 & = 2 \\
1111 & = 2
\end{align*} \]

\[ \begin{align*}
11 & \quad 131 = 4 \\
171 & = 4 \\
181 & = 2
\end{align*} \]

\[ \begin{align*}
12(10) & = \{1, 3, 7, 9, 11, 13, 17, 19, 3\} \\
14(60) & = 3 \\
111 & = 1 \\
131 & = 4 \\
171 & = 4 \\
181 & = 2 \\
1111 & = 2
\end{align*} \]

\[ \begin{align*}
12(12) & = 8 \\
161 & = 1 \\
11 & = 4 = 1r^3 \\
1r & = 6 \\
1r^2 & = 15r & = 15r^2 = 15r^3 = 2
\end{align*} \]

Other divide out of C! 

2. \[ \begin{align*}
\langle \frac{1}{3} \rangle & = \{\ldots, -3, -\frac{7}{3}, -2, -\frac{10}{3}, -1 - \frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots \} \text{ in } \mathbb{Q} \\
\langle \frac{1}{4} \rangle & = \{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots \} \text{ in } \mathbb{Q}^*
\end{align*} \]
4. Done in class

6. Suppose \( x^4 = e \) and \( x^2 = e \). If \( x^4 = e \) then \( x^{-4} = e \) so \( x^{4-2} x = e \).

Thus \( x^2 = e \).

If \( x^2 = e \) and \( x^5 = e \) then \( x^{\frac{9}{5}} = e \) so \( x^5 = e \).

Thus \( x^5 = e \).

The order of \( x \) must be \( \boxed{3} \) or \( 6 \).

7. Let \( |a| = n \). Then \( \{e, a, a^2, \ldots, a^{n-1}\} \) are all distinct, otherwise \( a^i = a^j \), \( i > j \Rightarrow a^{j-i} = e \), contradicting \( |a| = n \).

Thus we have \( n \) distinct elements of \( G \), so \( |a| = n \leq |G| \).

10. Let \( A \) be abelian, \( x, y \in A \) of order 2. Thus

\[ x^2 = e = y^2 \] so \( x = x^{-1}, y = y^{-1} \). Notice \( xy = x, xy = y \) by cancellation.

Thus \( \{e, x, y, xy\} \) is a subgroup of order 4.

With Cayley Table

\[
\begin{array}{c|cccc}
  & e & x & y & xy \\
\hline
e & e & x & y & xy \\
x & x & e & y & xy \\
y & y & x & e & x \\
xy & xy & y & e & x \\
\end{array}
\]