10. Clearly $A + B$ is closed under subtraction. For $a + b \in A + B$ and $r \in R$ then $r(a + b) = ra + rb \in A + B$ since $ra \in A$ and $rb \in B$ because $A$ and $B$ are ideals. Similarly $(a + b)r \in A + B$ so $A + B$ is an ideal.

34. $R/I$ is commutative iff $(r + I)(s + I) = (s + I)(r + I)$ for all $s, r \in R$. This holds iff $rs + I = sr + I$ which, by the rule for equality of cosets, is equivalent to $rs - sr \in I$ for all $s, r \in R$.

36. Let $x = 2 + 2i$. Notice that $ix = -2 + 2i$ so we immediately get that $4$ and $4i$ are in the ideal. It is easy to check that $2$ is not of the form $(a + bi)x$ for $a, b \in \mathbb{Z}$. Since $2 \equiv -2i$ in the quotient, we get that a complete set of coset representatives is $\{0, i, 2i, 3i, 1, 1+i, 1+2i, 1+3i\}$. The quotient has eight elements and has characteristic 4.

39. Let $I$ be a nonzero ideal in $\mathbb{Z}$. Notice that $a \in I$ iff $-a \in I$ so choose $n > 0$ minimal in $I$. Let $m \in I$, $m > 0$, so $m \geq n$. By the division algorithm we can write $m = qn + r$ with $0 \leq r < n$. But $m, qn \in I$ so $r \in I$ so $r$ must be zero by minimality of $n$. Thus $m = qn$ so $m \in (n)$. Thus $I = (n)$ is principal.

42. Let $r_1, r_2 \in N(A)$ and $r \in R$. So there exists $n_1, n_2$ such that $r_1^{n_1} \in A$. Consider $(r_1 - r_2)^{n_1 + n_2}$. Expanding this out by the binomial theorem (since $R$ is commutative!) we get a bunch of terms of the form $\pm r_1^a r_2^b$ where $a + b = n_1 + n_2$. Thus either $a \geq n_1$ or $b \geq n_2$ so each term is in $A$. Thus $r_1 - r_2 \in N(A)$. Also $(rr_1)^{n_1} = r^{n_1} r_1^{n_1} \in A$ by commutativity, and since $r_1^{n_1} \in A$. Thus $rr_1 \in N(A)$ and $N(A)$ is an ideal.

45. See back.

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9. See back.

27. Let $R$ be finite commutative ring with unity and $P$ a prime ideal. Then $R/P$ is an integral domain, necessarily finite. But finite integral domains are fields so $R/P$ is a field. Thus $P$ is a maximal ideal.

30. No. For example $\mathbb{Z}$ is an integral domain, $6\mathbb{Z}$ is a proper ideal and $\mathbb{Z}/6\mathbb{Z}$ is not an integral domain.

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7. Suppose $f : 2\mathbb{Z} \to 3\mathbb{Z}$ is an isomorphism and let $0 \neq a = f(2)$. Then $f(4) = f(2 + 2) = f(2) + f(2) = a + a = 2a$. But $f(4) = f(2 \cdot 2) = f(2)f(2) = a^2$. Thus $a^2 = 2a$ which implies $a = 2$ but $2$ is not in $3\mathbb{Z}$ so no isomorphism can exist. This proof works for $4\mathbb{Z}$ as well, $2\mathbb{Z}$ and $4\mathbb{Z}$ are not isomorphic rings. Notice that all three are isomorphic to $\mathbb{Z}$ as abelian groups.
41. a. See back.

b. Suppose \( A \) is maximal in \( S \), so \( S/A \) is a field. Consider the composition of ring homomorphisms below:

\[
R \xrightarrow{\phi} S \xrightarrow{\pi} S/A
\]

where \( \pi \) is the natural projection map. It is clear that \( \pi \circ \phi \) is onto with kernel just \( \phi^{-1}(A) \). So by the first isomorphism theorem \( R/\phi^{-1}(A) \cong S/A \) which is a field. Thus \( \phi^{-1}(A) \) is a maximal ideal.

60.a. Easy check.

b. The kernel is matrices of the form \( \begin{pmatrix} a & a \\ a & a \end{pmatrix} \).

c. \( \phi \) is clearly onto so this follows from the first isomorphism theorem.

d. Yes.

e. No, \( \mathbb{Z} \) is not a field.

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3. See back.

11. See back.

18. Suppose \( (x) \subset I \subset \mathbb{Q}[x] \). Let \( p(x) \in I - (x) \) so \( p(x) = a_0 + a_1x + \cdots + a_nx^n \) where \( a_0 \neq 0 \). But \( a_1x + \cdots + a_nx^n \in (x) \subset I \) so \( a_0 \in I \). But \( a_0 \) is a unit so \( I = \mathbb{Q}[x] \). Thus \( (x) \) is maximal.

20. Let \( p(x) = f(x) - g(x) \). The assumption is that \( p(a) = 0 \) for infinitely many \( a \), so \( p(x) \) has infinitely many roots. This is impossible by Cor.3 unless \( p(x) = 0 \), i.e. \( f(x) = g(x) \).

40. Let \( I = (x^2 - 2) \), so \( 2 + I = x^2 + I \). Using this we see that any element of \( \mathbb{Q}[x]/I \) can be expressed in the form \( a + bx + I \). However if \( a + bx + I = c + dx + I \) then \( (a-c) + (b-d)x \) is a multiple of \( (x^2 - 2) \) which can’t happen unless \( a = c \) and \( b = d \). Thus \( \{a + bx + I \} \) is a complete list of distinct cosets. Notice that:

\[
(a + bx + I)(c + dx + I) = ac + bdx^2 + (ad + bc)x + I = ac + 2bd + (ad + bc)x + I.
\]

But this multiplication is the “same” as multiplying \( (a + b\sqrt{2})(c + d\sqrt{2}) \) (just replace \( x \) by \( \sqrt{2} \) in the equation above), so the map \( a + bx + I \to a + b\sqrt{2} \) gives an isomorphism between the rings. (You need to check it’s an additive homomorphism as well, this is easy).