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10. Clearly $A + B$ is closed under subtraction. For $a + b \in A + B$ and $r \in R$ then $r(a + b) = ra + rb \in A + B$ since $ra \in A$ and $rb \in B$ because A and B are ideals. Similarly $(a + b)r \in A + B$ so $A + B$ is an ideal.

34. R/I is commutative iff $(r + I)(s + I) = (s + I)(r + I)$ for all $s, r \in R$. This holds iff $rs + I = sr + I$ which, by the rule for equality of cosets, is equivalent to $rs - sr \in I$ for all $s, r \in R$.

36. Let $x = 2 + 2i$. Notice that $ix = -2 + 2i$ so we immediately get that 4 and $4i$ are in the ideal. It is easy to check that 2 is not of the form $(a + bi)x$ for $a, b \in Z$. Since $2 \equiv -2i$ in the quotient, we get that a complete set of coset representatives is $\{0, i, 2i, 3i, 1, 1+i, 1+2i, 1+3i\}$. The quotient has eight elements and has characteristic 4.

39. Let I be a nonzero ideal in Z. Notice that $a \in I$ iff $-a \in I$ so choose $n > 0$ minimal in I. Let $m \in I$, $m > 0$, so $m \geq n$. By the division algorithm we can write $m = qn + r$ with $0 \leq r < n$. But $m, qn \in I$ so $r \in I$ so r must be zero by minimality of n. Thus $m = qn$ so $m \in (n)$. Thus $I = (n)$ is principal.

42. Let $r_1, r_2 \in N(A)$ and $r \in R$. So there exists n_i such that $r_i^{n_i} \in A$. Consider $(r_1 - r_2)^{n_1+n_2}$. Expanding this out by the binomial theorem (since R is commutative!) we get a bunch of terms of the form $\pm r_1^a r_2^b$ where $a + b = n_1 + n_2$. Thus either $a \geq n_1$ or $b \geq n_2$ so each term is in A. Thus $r_1 - r_2 \in N(A)$. Also $(rr_1)^{n_1} = r^{n_1} r_1^{n_1} \in A$ by commutativity, and since $r_1^{n_1} \in A$. Thus $rr_1 \in N(A)$ and $N(A)$ is an ideal.

45. See back.

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9. See back.

27. Let R be finite commutative ring with unity and P a prime ideal. Then R/P is an integral domain, necessarily finite. But finite integral domains are fields so R/P is a field. Thus P is a maximal ideal.

30. No. For example Z is an integral domain, $6Z$ is a proper ideal and $Z/6Z$ is not an integral domain.

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7. Suppose $f : 2Z \to 3Z$ is an isomorphism and let $0 \neq a = f(2)$. Then $f(4) = f(2 + 2) = f(2) + f(2) = a + a = 2a$. But $f(4) = f(2 * 2) = f(2)f(2) = a^2$. Thus $a^2 = 2a$ which implies $a = 2$ but 2 is not in $3Z$ so no isomorphism can exist. This proof works for $4Z$ as well, $2Z$ and $4Z$ are not isomorphic rings. Notice that all three are isomorphic to Z as abelian groups.

41. a. See back.

b. Suppose A is maximal in S, so S/A is a field. Consider the composition of ring homomorphisms below:

$$R \xrightarrow{\phi} S \xrightarrow{\pi} S/A$$

where π is the natural projection map. It is clear that $\pi \circ phi$ is onto with kernel just $\phi^{-1}(A)$. So by the first isomorphism theorem $R/\phi^{-1}(A) \cong S/A$ which is a field. Thus $\phi^{-1}(A)$ is a maximal ideal.

60.a. Easy check.

b. The kernel is matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$.

c.ϕ is clearly onto so this follows from the first isomorphism theorem.

d. Yes.

e. No, Z is not a field.

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3. See back.

11. See back.

18. Suppose $(x) \subset I \subseteq Q[x]$. Let $p(x) \in I - (x)$ so $p(x) = a_0 + a_1x + \cdots + a_nx^n$ where $a_0 \neq 0$. But $a_1x + \cdots + a_nx^n \in (x) \subset I$ so $a_0 \in I$. But a_0 is a unit so $I = Q[x]$. Thus (x) is maximal.

20. Let $p(x) = f(x) - g(x)$. The assumption is that $p(a) = 0$ for infinitely many a, so $p(x)$ has infinitely many roots. This is impossible by Cor.3 unless $p(x) = 0$, i.e. $f(x) = g(x)$.

40. Let $I = (x^2 - 2)$, so $2 + I = x^2 + I$. Using this we see that any element of $Q[x]/I$ can be expressed in the form $a + bx + I$. However if $a + bx + I = c + dx + I$ then $(a - c) + (b - d)x$ is a multiple of $(x^2 - 2)$ which can't happen unless $a = c$ and $b = d$. Thus $\{a + bx + I\}$ is a complete list of distinct cosets. Notice that:

$$(a + bx + I)(c + dx + I) = ac + bdx^2 + (ad + bc)x + I = ac + 2bd + (ad + bc)x + I.$$

But this multiplication is the "same" as multiplying $(a + b\sqrt{2})(c + d\sqrt{2}$ (just replace x by $\sqrt{2}$ in the equation above), so the map $a + bx + I \to a + b\sqrt{2}$ gives an isomorphism between the rings. (You need to check it's an additive homomorphism as well, this is easy).