Math 419 Midterm Exam #1 - October 5, 2007

1. (20 points) Complete the following:

a. The alternating group is …
   \[ S_n \]
   \[ \text{The set of even permutations in } S_n \]

b. Let \( g \in G \). The centralizer of \( g \) …
   \[ C_G(g) = \{ x \in G | xg = gx \} \]

c. Lagrange’s Theorem states that …
   \[ \text{Let } H \leq G, \ |H| < \infty. \text{ Then} \]
   \[ |H| \mid |G| \]

d. The order of an element \( g \) in a group \( G \) is …
   \[ \text{The smallest } n \in \mathbb{N}^+ \text{ such that } g^n = e. \]
   \[ \text{If none exists, } g \text{ has infinite order.} \]

2. (10 points) Give a subgroup of \( D_4 \) of order 4 which is not cyclic.

\[ \{ e, s, r^2, sr^2 \} \]

3. (10 points) Write down an element of \( S_{12} \) of order 60. Is it in \( A_{12} \)? Explain.

\[ o = (1 2 3)(1 4 5 6 7)(1 8 9 10 11 12) \]

\[ \text{Not in } A_{12} \]

\[ (1 2 3) \text{ and } (1 8 9 10 11 12) \text{ are even} \]

\[ (4 5 6 7) \text{ is odd} \]

\[ \text{So } o \text{ is odd} \]
4. (15 points) True or false:

- a. All cyclic groups are finite.  \[ \mathbb{Z} \]

- b. Matrix multiplication is commutative.  \[ \text{False} \]

- c. Every finite group is isomorphic to a subgroup of some symmetric group.  \[ \text{Cayley} \]

- d. All nonabelian groups of order 14 are isomorphic.  \[ \text{All } \cong \text{ } \mathbb{Z}_7 \times \mathbb{Z}_2 \]

- e. The intersection of two subgroups of a group is always a subgroup.  \[ \text{True} \]

- f. The union of two subgroups of a group is always a subgroup.  \[ \text{False} \]

- g. A group of order 12 must contain an element of order 6.  \[ \text{e.g. } A_4 \]

- h. The group \( \mathbb{Z}/12\mathbb{Z} \) has four generators.  \[ 1, 5, 7, 11 \]

- i. The set of odd permutations in \( S_n \) form a single left coset of \( A_n \) in \( S_n \).  \[ \text{True} \]

- j. 12 and 42 generate the same cyclic subgroup of \( \mathbb{Z}/90\mathbb{Z} \).

Both have gcd of 6 with 90.
5. **(10 points)** Draw the subgroup lattice for the group \(\mathbb{Z}/18\mathbb{Z}\). Each subgroup should be accompanied by a list of its elements.

\[
\mathbb{Z}/18\mathbb{Z} = \langle 0, 1, 2, 3, 4, \ldots, 17 \rangle
\]

\[
\langle 2 \rangle = \langle 0, 2, 4, 6, 8, 10, 12, 14, 16 \rangle
\]

\[
\langle 3 \rangle = \langle 0, 3, 6, 9, 12, 15 \rangle
\]

\[
\langle 6 \rangle = \langle 0, 6, 12 \rangle
\]

\[
\langle 9 \rangle = \langle 0, 9 \rangle
\]

\[
\{ 0 \}
\]

6. **(15 points)** Let \(\sigma = (1, 2, 3)(2, 3, 5)(4, 6, 7)(1, 3, 2)(6, 7)\)

a. Write \(\sigma\) and \(\sigma^{-1}\) in disjoint cycle notation.

\[
\sigma = (1 \ 5 \ 3)(2 \ 4 \ 6)
\]

\[
\sigma^{-1} = (1 \ 3 \ 5)(4 \ 6)
\]

b. Find the order of \(\sigma\).

\[6\]

c. Is \(\sigma\) even or odd?

\[\text{odd}\]

d. Write down \(\sigma^{35}\) in disjoint cycle notation.

\[
\sigma^{-6} = e \Rightarrow \sigma^{-33} = \sigma^{-3} = (1 \ 5 \ 3)(4 \ 6)
\]

\[
= (4 \ 6)
\]
7. (20 points) A subgroup \( H \leq G \) is said to be characteristic if \( \phi(H) = H \) for all automorphisms \( \phi \) of \( G \).

a. Define the center \( Z(G) \) of a group and prove it is a subgroup.

b. Prove that the center is characteristic.

\[ Z(G) = \{ z \in G | zg = gz, \forall g \in G \} = Z \]

(Prove \( e \in Z(G) \). Let \( z, z' \in Z \). \( z, z'g = zg, z'g \) since \( g \in Z \).

So \( zg = z'g \).

Let \( g \in G \). \( g^{-1}zg = g^{-1}g \) since \( z \in Z \).

\( z'g = zg \) by taking inverses, so \( z' \in Z \).

Thus \( Z \leq G \).

b. Let \( \phi \in \text{Aut}(G) \). Let \( g \in G \). Then \( g = \phi(g) \) since \( \phi \) is an \( \text{Aut} \).

Let \( z \in Z(G) \). \( \phi(z)g = \phi(z) \phi(g) \)

\( = \phi(zg) \)

\( = \phi(gz) \) since \( z \in Z \)

\( = \phi(g) \phi(z) = g \phi(z) \) so \( \phi(z) \in Z(G) \).

Thus \( \phi(Z) \leq Z \).

Now let \( z \in Z(G) \). \( z = \phi(z) \) since \( \phi \) is onto. Let \( g \in G \).

\( \phi(gz) = \phi(g \phi(z)) = \phi(g) \phi(z) = z \phi(g) \) since \( z \in Z(G) \)

\( = z \phi(g) \) since \( \phi(z) \in Z(G) \).

But \( \phi \) is 1-1 so \( gz = zg \) so \( z \in Z(G) \).

Thus \( \phi \) maps \( Z(G) \) onto \( Z(G) \).

Thus \( Z(G) \) is characteristic.