Solutions

Math 353 Midterm Exam - October 17, 2014

Instructions: You may not use any notes, books, calculators, etc... It is ok if your final answers include binomial coefficients and if you do not multiply out exponentials.

1. (55 points) Short answer, little or partial credit.
   a. Define a derangement.

   A permutation with no fixed points,
   \[ i.e. \sigma(i) \neq i \quad \forall i. \]

   b. A class of 20 students wishes to elect a president, vice president, and three senators. How many ways can this be done?

   \[ 20 \cdot 19 \cdot \binom{18}{3} \]

   c. A five-card poker hand is dealt. What is the probability of getting two-pair? (for example AATT3 is two-pair but AAATT is not, it is a full-house.)

   \[
   \binom{13}{2} \binom{4}{2} \binom{4}{2} \cdot \frac{4 \cdot 4}{\binom{52}{5}}
   \]

   d. Let \( \sigma = (123)(45)(678) \) and \( \tau = (1568)(234)(7) \) be elements of the symmetric group \( S_8 \). Write the products \( \sigma \tau \) and \( \sigma^2 \) in cycle notation.

   \[ \sigma \tau = (1435782)(16) \]

   \[ \sigma^2 = (132)(1678) (4)(5) \]
e. Let $S(n, k)$ denote the Stirling number of the second kind. Calculate $S(5, 2)$.

5 distinct objects, 2 boxes, no empty

$4 + 1$ 5 choices

$3 + 2$ (5) = 10 choices

15

f. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 15$ where the $x_i$ are nonnegative integers?

15 dots 3 plus signs

\begin{pmatrix} 18 \\ 3 \end{pmatrix}

g. How many triangulations are there of a octagon?

An $n$-gon has $C_n$ triangulations.

$C_8 = \binom{13}{2} = \frac{13!}{2!(13-2)!} = \frac{13 	imes 12}{2} = 132$

132

h. Calculate $p(8, 3)$, the number of partitions of 8 with less than or equal to three parts.

8
7 1
6 2
6 1 1
5 3
5 2 1
4 4
4 3 1
4 2 2
3 3 2

10
i. A fair coin is tossed 6 times. What is the probability of getting at least 4 heads?

\[ \frac{\binom{6}{4}}{2^6} + \frac{\binom{6}{5}}{2^6} + \frac{\binom{6}{6}}{2^6} = \frac{22}{64} = \frac{11}{32} \]

j. A biased coin comes up heads with probability 0.8. If it is tossed 6 times what is the probability of getting at least 4 heads.

4 heads: \(( \binom{6}{4} 1.8^4 0.2^2)\) 5 heads: \(( \binom{6}{5} 1.8^5 0.21)\) 6 heads: \(( \binom{6}{6} 1.8^6)\)

\[
\binom{6}{4} (1.8^4)(0.2^2) + \binom{6}{5} (1.8^5)(0.21) + \binom{6}{6} (1.8^6)
\]

k. Find the generating function for the sequence \(a_n = n\).

\[ \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \]

\[ \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \]

\[ \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots = \sum n x^n \]

\[ \frac{x}{(1-x)^2} \]
2. (15 points) A bag contains 3 red, 2 blue and 6 green marbles. A marble is sampled and replaced 7 times. Find the probability that all three colors have already been seen.

Let $M_R = \#$ ways to have no reds. Similarly $M_B, M_G$.

Total of $11^7$ draws need to subtract $|M_R \cup M_B \cup M_G|$.

$|M_R \cup M_B \cup M_G| = |M_R| + |M_B| + |M_G| - |M_R \cap M_B| - |M_R \cap M_G| - |M_B \cap M_G| + |M_R \cap M_B \cap M_G|$

$= 8^7 + 9^7 + 5^7 - 6^7 - 2^7 - 3^7 + 0$

Prob $= \frac{11^7 - 8^7 - 9^7 - 5^7 + 6^7 + 2^7 + 3^7}{11^7}$
3. (15 points) a. Define the Stirling number of the second kind, \( S(n, k) \).

b. Give a combinatorial proof of the identity: \( S(n+1, k+1) = \sum_{i=k}^{n} \binom{i}{k} S(i, k) \).

a. The \# of ways to assign \( n \) distinct balls into \( k \) identical boxes with no empty boxes.

b. Fix a ball \( b \). Divide up the possible assignments based on the \# of balls in the remaining \( k \) boxes with \( b \). There must be at least \( k \) since each box has \( \geq 1 \). There are at most \( n \) if \( b \) is alone.

The \# of ways to assign \( n \) \( b \) balls not in \( b \)'s box is

\[
\binom{n}{k} \cdot S(n, k)
\]

\( \binom{n}{k} \) choose balls not with \( b \)
\( S(n, k) \) assign to \( k \) boxes no empty

All assignments of \( n+1 \) balls to \( k+1 \) boxes are counted once.

Thus

\[
S(n+1, k+1) = \binom{n}{k} S(k, k) + \binom{n}{k+1} S(k+1, k+1) + \ldots + \binom{n}{k+1} S(k+1, k+1)
\]

as desired.
4. (15 points) Consider the sequence given by $a_1 = 8, a_2 = 46$ and $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 3$. Solve the recursion to find a formula for $a_n$.

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

Polynomial is $x^2 - 7x + 10 = (x-5)(x-2)$

General solution $A \cdot 5^n + B \cdot 2^n$

$8 = A \cdot 5 + 2 \cdot B$

$46 = A \cdot 25 + 4 \cdot B$

$\Rightarrow 30 = 15A \Rightarrow A = 2$

$\Rightarrow 13 = 10B \Rightarrow B = -1$

$$a_n = 2 \cdot 5^n - 1 \cdot 2^n$$