1. 12.2.2B
The integers \( \mathbb{Z} \) are acting on \( G \) by \( n \triangleright g = g_0^n g \). So \( 0 \triangleright g = g_0^0 g = eg = g \) and the first axiom is satisfied.
Let \( n_1, n_2 \in \mathbb{Z} \). Then :

\[
n_1 \triangleright n_2 \triangleright g = n_1 \triangleright g_0^{n_2} g = g_0^{n_1} g_0^{n_2} g = g_0^{n_1 + n_2} g = (n_1 + n_2) \triangleright g
\]
so the second axiom is satisfied.

2. Let \( H \) and \( K \) be two subgroups of a group \( G \). Prove that their intersection \( H \cap K \) is also a subgroup. For extra credit prove that the union \( H \cup K \) is never a subgroup except in the trivial situation where \( H \subseteq K \) or \( K \subseteq H \).

Let \( x, y \in H \cap K \). Since \( H \subseteq G \) we know \( x^{-1} \) and \( xy \) are in \( H \). Since \( K \subseteq G \) we know \( x^{-1} \) and \( xy \) are in \( K \). Thus \( x^{-1} \) and \( xy \) are in \( H \cap K \) and so \( H \cap K \) is a subgroup.

For the extra credit suppose \( H \) and \( K \) are subgroups and neither \( H \subseteq K \) nor \( K \subseteq H \). We must show \( H \cup K \) is not a subgroup. By our assumption we can choose \( h \in H \) with \( h \notin K \). Also choose \( k \in K \) with \( k \notin H \). So \( h, k \in H \cup K \) and we will show \( h k \notin H \cup K \). If \( h k = h' \in H \) then \( k = h^{-1} h' \in H \), a contradiction. Similarly if \( h k = k' \in K \) then \( h = k^{-1} k' \in K \) a contradiction. Thus \( h k \) is in neither \( h \) nor \( K \), so not in \( H \cup K \). Thus \( H \cup K \) is not closed under multiplication, so is not a subgroup.

3. Let \( G \) be a group and \( g \in G \). Define the centralizer of \( g \), denoted \( C_G(g) \), as the elements that commute with \( g \), namely:

\[
C_G(g) = \{ x \in G \mid xg = gx \}.
\]

a. Prove that \( C_G(g) \) is a subgroup of \( G \).

b. Let \( \sigma = (1, 2)(3, 4) \in S_4 \). Calculate \( C_{S_4}(\sigma) \).

c. Let \( A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}) \). Calculate the centralizer of \( A \).

d. Describe the center \( Z(G) \) in terms of centralizers.

3a. First observe \( eg = ge = g \) so \( e \in C_G(g) \). Now suppose \( x, y \in C_G(g) \) so \( xg = gx \) and \( yg = gy \) by definition. Then \( xyg = xgy = gxy \) so \( xy \in C_G(g) \). Take the equation \( xg = gx \) and multiply both sides by \( x^{-1} \) on the left and on the right we get: \( gx^{-1} = x^{-1}g \) so \( x^{-1} \in C_G(g) \). Thus \( C_G(g) \) is closed under multiplication and taking inverses so \( C_G(g) \leq G \).

3b. \( C_{S_4}(\sigma) = \{ e, (1, 2), (3, 4), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3), (1, 3, 2, 4), (1, 4, 2, 3) \} \). Notice this centralizer is isomorphic to \( D_8 \).

c. The matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in the centralizer of \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) if and only if it is invertible and:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Multiplying out we get:

\[
\begin{pmatrix} a & a + b \\ c & c + d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix}.
\]

This gives us 4 equations which we solve to show that \( c = 0 \) and \( a = d \). So the centralizer is:

\[
\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}
\]
where the condition on $a$ ensures the matrix is invertible.

d. The center of $G$ is the intersection of the centralizers of the elements of $G$.

4. Calculate the conjugacy classes in the dihedral group $D_8$. Repeat for $D_{10}$.

For $D_8$ you should get:

$$\{e\}, \{r, r^3\}, \{r^2\}, \{s, sr^2\}, \{sr, sr^3\}.$$

For $D_{10}$ you should get:

$$\{e\}, \{r, r^4\}, \{r^2, r^3\}, \{s, sr, sr^2, r^3, sr^4\}.$$

Notice all 5 reflections are conjugate for the symmetries of a pentagon whereas for the square there are two conjugacy classes. Can you see why geometrically?

5. 12.3.2A

See back of book.

6. 12.4.1B

i. To get an element of $X$ we can choose anything we like for $(g_1, g_2, g_3, \ldots, g_{p-1})$. Once we do this our choice of $g_p$ is forced on us, since we need $g_1g_2 \cdots g_p = e$ then we must choose $g_p = g_{p-1}^{-1}g_{p-2}^{-1} \cdots g_1^{-1}$. Thus $X$ has $|G|^{p-1}$ elements.

ii. It is clear that $$0 \triangleright (g_1, g_2, g_3, \ldots, g_p) = (g_1, g_2, g_3, \ldots, g_p) = p \triangleright (g_1, g_2, g_3, \ldots, g_p)$$
so the action of $Z_p$ is well-defined. One easily checks that $$a \triangleright b \triangleright (g_1, g_2, g_3, \ldots, g_p) = (a + b) \triangleright (g_1, g_2, g_3, \ldots, g_p).$$

Rotating by $a$ and then by $b$ is the same as rotating by $a + b$. Finally we need to check that the rotated tuples are still in $X$. Multiply the equation

$$g_1g_2 \cdots g_p = e$$

by $g_1^{-1}$ on the left and right to get:

$$g_2g_3 \cdots g_pg_1 = e.$$ 

Repeating with $g_2$ etc... shows us that all the cyclic permutations remain in $X$.

If any $g_i \neq g_j$ then rotating by $j - i$ will move $g_i$ into the $j$ position and so we will have a different tuple. Thus the only elements fixed by all of $Z_p$ are tuples of the form $(g, g, g, \ldots, g)$.

iii. We know from the orbit stabilizer theorem that all the orbits have order dividing the order of $Z_p$. We know from part i that $X$ is a multiple of $p$. Since we have an orbit $(e, e, \ldots, e)$ of size 1, there must be at least $p - 1$ other orbits of size 1. But orbits of size one are tuples $(g, g, \ldots, g)$ with $g^p = e$. Thus $G$ has at least $p - 1$ elements of order $p$.

7. Let $G = S_4$ be the symmetric group on 4 letters. Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$ and let $K = \{e, (12), (34), (12)(34)\}$. Verify that $H$ and $K$ are both subgroups of $S_4$ and both are isomorphic to the Klein 4 group. Next compute the left and right cosets of $H$. Repeat for $K$. What do you notice?
Left and right cosets of $H$ are the same:

\[
\begin{align*}
e H &= He = \{e, (12)(34), (13)(24), (14)(23)\} \\
(12)H &= H(12) = \{(12), (34), (1324), (1423)\} \\
(13)H &= H(13) = \{(13), (1234), (24), (1432)\} \\
(14)H &= H(14) = \{(14), (1243), (1342), (23)\} \\
(123)H &= H(123) = \{(123), (134), (243), (142)\} \\
(124)H &= H(124) = \{(124), (143), (132), (234)\}
\end{align*}
\]

Left cosets of $K$ are:

\[
\begin{align*}
e K &= \{e, (12), (34), (12)(34)\} \\
(13)K &= \{(13), (123), (134), (1234)\} \\
(14)K &= \{(14), (124), (143), (1243)\} \\
(24)K &= \{(24), (142), (243), (1432)\} \\
(23)K &= \{(23), (132), (234), (1342)\} \\
(13)(24)K &= \{(13)(24), (1423), (1324), (14)(23)\}
\end{align*}
\]

Right cosets of $K$ are:

\[
\begin{align*}
Ke &= \{e, (12), (34), (12)(34)\} \\
K(13) &= \{(13), (132), (143), (1432)\} \\
K(23) &= \{(23), (123), (243), (1243)\} \\
K(24) &= \{(24), (124), (234), (1234)\} \\
K(14) &= \{(14), (142), (134), (1342)\} \\
K(13)(24) &= \{(13)(24), (1324), (1423), (14)(23)\}
\end{align*}
\]