1. $\sigma \tau = (1,3,5,4,2)(6,8,10,12,7)(9,11), \tau \sigma = (1,3,2,4,6)(5,7,9,11,8)(10,12)$.  
$\tau^{-1} = (1,4,3,2)(5,7,6)(8,12,11,10,9)\tau \sigma \tau^{-1} = (2,3,4)(1,6,7)(5,9)(10,11,12,8)$.  
The order of an $n$ cycle is $n$. The permutation $\sigma$ has order 12 and $\tau$ has order 60.  

   In one-line notation we have $\sigma = 2,3,1,5,6,4,8,7,10,11,12,9$ which is not 231-avoiding.  

2. There are 4 elements in our group so the maximum order of an element is 4. If $g$ has order 4 then $\{e,g,g^2,g^3\}$ are all distinct so our group $G$ is cyclic of order 4. Suppose we have an element $x$ of order 3, so we can write $G = \{e,x,x^2,y\}$. The submatrix of the Cayley table from $\{e,x,x^2\}$ already has a $e,x,x^2$ in each row and column, so there is no way to fill in the row for $y$ and no such group exists. (Or use Lagrange’s theorem to rule out this case). Finally we come to the case where all nonidentity elements have order 2 so let $x \neq y$ have order two. Then $xy$ is not equal to $x$ or $y$ by cancellation so it also has order two. Thus we have $G = \{e,x,y,xy\}$ with $x^2 = y^2 = (xy)^2 = e$. Use this equation to show $xy = yx$ so we have the Klein 4 group.

3. (11.3.2B) $G$ has only 4 elements of order 1 or 2 so any hypothetical subgroup must contain at least one 3-cycle, and it’s inverse. However it is easy to check that $\{e,(12)(34),(13)(24),(14)(23),(abc),(acb)\}$ is not closed under multiplication. So we have at least two different pairs of 3cycles, say $(a,b,c)$ and $(a,b,d)$ without loss of generality. Multiplying these in both orders gives $(ac)(bd)$ and $(ad)(bc)$ so we end up with already 4 cycles, 2 elements of order 2 and the identity. Too big! Thus $G$ has no subgroup of order 6.

4. (11.4.1B) For any of the 4 corners of a tetrahedron one can fix that corner and rotate the opposite triangle by 120 or 240 degrees, so this gives 8 symmetries of order 3. There are also 3 rotations of 180 degrees which have order 2. Allowing orientation reversing you also get reflections of order 2 for a total of 24 symmetries.

5. (11.5.3B) To find an element of maximal order in $S_n$ we need to find the partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_t)$ of $n$ that maximizes the gcd of $\{\lambda_1,\lambda_2,\ldots,\lambda_t\}$. Then any permutation of that cycle type will work.

<table>
<thead>
<tr>
<th>n</th>
<th>sigma</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>e</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(12)</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>(123)</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>(1234)</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>(123)(45)</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>(123)(45)</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>(1234)(567)</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>(12345)(678)</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>(12345)(6789)</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>(12345)(678)(9,10)</td>
<td>30</td>
</tr>
</tbody>
</table>

6. The following elements have order 12: $\{1,5,7,11\}$.  
The following elements have order 6: $\{2,10\}$.  
The following elements have order 4: $\{3,9\}$.  
The following elements have order 3: $\{4,8\}$. 
The following elements have order 2: \{6\}. And \{0\} has order 1.

7. The matrix of a 120 degree rotation has order 3, so for example \(\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\). The matrix \(A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) has the property that \(A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\), so clearly has infinite order.

8. Let \(z_1, z_2 \in Z(G)\) and let \(g \in G\). Then:

\[
(z_1 z_2)g = z_1 (gz_2)\quad \text{by associativity and since } z_2 \text{ is in the center.}
\]

\[
= g (z_1 z_2)\quad \text{by associativity and since } z_1 \text{ is in the center.}
\]

Thus \(z_1 z_2 \in Z(G)\) so \(Z(G)\) is closed under the operation. Now suppose \(z \in Z(G)\) and let \(g \in G\). Then since \(z\) is in the center we get:

\[
z g^{-1} = g^{-1} z.
\]

Inverting both sides of this equation gives us:

\[
g z^{-1} = z^{-1} g
\]

so \(z^{-1}\) is in the center. Thus \(Z(G) \leq G\).