1. **5.1.6B** Let’s define $X(n, k)$ to be all the possible products of $n - k$ integers taken from \{1, 2, \ldots, k\}, repeats allowed. We must show that for $1 \leq k < n$ that

$$\sum_{a \in X(n, k)} a = S(n, k).$$

We proceed by induction on $n$. It’s easy to see that $X(2, 1) = \{1\}$ and $S(2, 1) = 1$ so the base case holds.

Now divide $X(n, k)$ into two subsets. The first, call it $A$, is the products that do not include the integer $k$. The second, call it $B$, are the products including $k$. Now $A$ is all products of $n - k$ integers taken from \{1, 2, \ldots, k - 1\} so by induction the elements in $A$ add up to $S(n - 1, k - 1)$. Each element of $B$ is of the form $k \cdot y$ where $y$ is a product of $n - k - 1$ integers taken from \{1, 2, \ldots, k\}. So by induction the elements in $B$ add up to $kS(n - 1, k)$. Since $X(n, k) = A \cup B$ we have:

$$\sum_{a \in X(n, k)} a = S(n - 1, k - 1) + kS(n - 1, k)$$

which equals $S(n, k)$ by Theorem 3.4.

2. **5.2.3B** (typo in the book here, should be $s(n, r)$ not $S(n, r)$.)

The Stirling number $s(n + 1, k + 1)$ is the absolute value of the coefficient of $x^{k+1}$ in $[x]_{n+1}$. Cancelling the first $x$ we see it is the coefficient of $x^k$ in $(x - 1)(x - 2) \cdots (x - n)$. Now do the substitution $y = x - 1$. Then:

$$(x - 1)(x - 2) \cdots (x - n) = y(y - 1)(y - 2) \cdots (y - n + 1) = [y]_n.$$

Thus $s(n + 1, k + 1)$ is the coefficient of $x^k$ in $[y]_n$. However by definition we have

$$[y]_n = \sum_{r=1}^{n} (-1)^{r-1} s(n, r) y^r = \sum_{r=1}^{n} (-1)^{r-1} s(n, r) (x - 1)^r.$$

Notice there is no $x^k$ term in (??) until $r = k$ and the coefficient of $x^k$ in $(x - 1)^r$ is $(-1)^{r-k} \binom{r}{k}$, by the binomial theorem. So the coefficient of $x^k$ in (?) is:

$$\sum_{r=k}^{n} (-1)^{r-1} s(n, r)(-1)^{r-k} \binom{r}{k} = (-1)^{1-k} \sum_{r=k}^{n} s(n, r) \binom{r}{k}.$$

So taking absolute value we get:

$$s(n + 1, k + 1) = \sum_{r=k}^{n} \binom{r}{k} s(n, r)$$

as desired. **Note:** I have not been able to find a combinatorial proof but there must be one!

3. This is an easy induction proof, $d_0 = C_0 = 1$. Now suppose $d_i = C_i$ for $i < n$. Then:

$$d_n = \sum_{k=1}^{n} d_{k-1}d_{n-k} = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

by the inductive hypothesis. But the latter sum is just $C_n$ since $C_n$ satisfies the same recursion formula.

4. a. In one-line notation the 231 avoiding permutations in $S_4$ are:

$$\{1234, 1324, 2134, 3124, 3214, 1243, 2143, 1423, 1432, 4123, 4132, 4213, 4312, 4321\}.$$
b. Let $A_n$ be the number of 231-avoiding permutations in $S_n$. Let $1 \leq k \leq n$ and consider the set of 231-avoiding permutations in $S_n$ with $\sigma(k) = n$. So in one line notation $\sigma$ looks like:

$$??????n\ast\ast\ast\ast\ast\ast\ast$$

where there are $k-1$ question marks and $n-k$ stars. The question marks must represent the numbers $1, 2, \ldots, k-1$ and the stars are $k, k+1, \ldots, n-1$, because as soon as a number $\leq k-1$ appears to the right of $n$ then a number $\geq k$ will be to the left and the forbidden 231 pattern appears.

Thus $\sigma$ is 231 avoiding if the question marks form a 231 avoiding permutation of $\{1, 2, \ldots, k-1\}$ and the stars form a 231 avoiding permutation of $\{k, k+1, \ldots, n-1\}$. There are $A_{k-1}A_{n-k}$ choices (where $A_0$ is defined to be 1). This proves that:

$$A_n = \sum_{k=1}^{n} A_{k-1}A_{n-k}.$$ 

Since $A_0 = A_1 = 1$, the previous problem proves $A_n = C_n$.

5. 5.3.2B. Hints below.

- First use the recursion to prove that $C_n > n + 2$ for $n > 3$.
- Next prove from the definition that $(n+2)C_{n+1} = (4n+2)C_n$.
- Suppose $C_n$ is prime. Prove that $C_n$ divides $C_{n+1}$.
- Prove that $n$ must then be $\leq 4$.

**Proof:** We have the recursion:

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0.$$ 

The RHS has $n$ terms and once $n > 3$ we know two of them at least are $> 1$. This implies that for $n > 3$ that $C_n > n + 2$.

Now from the definition $C_n = \binom{2n}{n+1}$ it follows that:

$$C_{n+1} = \frac{\binom{2n+2}{n+1}}{n+2} = \frac{(2n+2)!}{(n+1)!(n+1)!n+2} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!(n+2)} = \frac{(4n+2)(2n!)}{(n+1)(n+2)n!n!} = \frac{4n+1}{n+2}C_n$$

Now assume $C_n$ is prime. From above we have that $(n+2)C_{n+1} = (4n+2)C_n$. Since we know $C_n > n+2$, then $C_n$ must divide $C_{n+1}$ since it cannot divide $n+2$. So $C_{n+1} = kC_n$ for some $k$. This gives:

$$4n+2 = k(n+2)$$

which is impossible if $k \geq 4$. So $k = 1, 2, 3$ which forces $n$ to be $\leq 4$. Thus the only prime Catalan numbers are $C_2$ and $C_3$.

6. See back of the book