1. 5.1.6B (Hint: Use Theorem 3.4 and proceed by induction on $n$). Let $X(n, k)$ be all possible products of $n - k$ integers from $\{1, 2, \ldots, k\}$, repeats allowed. Divide $X(n, k)$ into two subsets.

2. 5.2.3B (typo in the book here, should be $s(n, r)$ not $S(n, r)$.)

3. We now prove the Catalan numbers are uniquely determined by the recursion above. Suppose $\{d_n \mid n \geq 0\}$ is a sequence such that $d_0 = 1$ and:

$$d_n = \sum_{k=1}^{n} d_{k-1}d_{n-k}.$$ 

Prove by induction that $d_n = C_n$.

4. Suppose $w = w_1w_2\cdots w_n$ is a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation. Saw that $w$ is 231-avoiding if there do not exist indices $i < k < p$ such that $w_p < w_i < w_k$.

   a. Write down the 231 avoiding permutations in $S_4$. (Hint: There are 14 of them)

   b. Let $S_n^{231}$ be the set of 231-avoiding permutations in $S_n$. Prove that the size of $S_n^{231}$ satisfies the recursion and conclude from problem 4 that $\#(S_n^{231}) = C_n$. Hint: For an arbitrary 231-avoiding permutation $w$, consider the position of the letter $n$ in $w$.

5. 5.3.2B. Hints below.

   - First use the recursion to prove that $C_n > n + 2$ for $n > 3$.
   - Next prove from the definition that $(n + 2)C_{n+1} = (4n + 2)C_n$
   - Suppose $C_n$ is prime. Prove that $C_n$ divides $C_{n+1}$.
   - Prove that $n$ must then be $\leq 4$.

6. 5.3.8A