Math 353 Fall 2016 - Homework #3 Solutions

3.1.5B We have an odd number of balls in each cup, so let these numbers be $2x + 1, 2y + 1, 2z + 1$ and $2w + 1$ where $x, y, z, w \geq 0$. We need to count solutions to

$$2x + 1 + 2y + 1 + 2z + 1 + 2w + 1 = 20$$

with $x, y, z, w \geq 0$. These are the same as solutions to $x + y + z + w = 8$, so there are $\binom{11}{3} = 165$ solutions.

3.3.2B Done in class.

3. The $n = 0$ case is clear. We have:

$$B_0 = S(0, 0)$$
$$B_1 = S(1, 0) + S(1, 1)$$
$$B_2 = S(2, 0) + S(2, 1) + S(2, 2)$$
$$B_3 = S(3, 0) + S(3, 1) + S(3, 2) + S(3, 3)$$
$$B_4 = S(4, 0) + S(4, 1) + S(4, 2) + S(4, 3) + S(4, 4)$$
$$\cdots$$
$$B_{n-1} = S(n-1, 0) + S(n-1, 1) + S(n-1, 2) + S(n-1, 3) + \cdots + S(n-1, n-1)$$

Now multiply the equation for $B_j$ by $\binom{n-1}{j}$ and add up to get $\sum_{j=0}^{n-1} B_j$. Rather than computing the sum of the right hand sides row by row we do it column by column. The first column contributes:

$$S(0, 0) \binom{n-1}{0} + S(1, 0) \binom{n-1}{1} + \cdots S(n-1, 0) \binom{n-1}{n-1}$$

which equals $S(n, 1)$ by 3.3.2B. The second column contributes:

$$S(1, 1) \binom{n-1}{1} + S(2, 1) \binom{n-1}{2} + \cdots S(n-1, 1) \binom{n-1}{n-1}$$

which equals $S(n, 2)$ by 3.3.2B. The penultimate column contributes:

$$S(n-2, n-2) \binom{n-1}{n-2} + S(n-1, n-2) \binom{n-1}{n-1}$$

which equals $S(n, n-1)$ by 3.3.2B

And the final column is just $S(n-1, n-1) \binom{n-1}{n-1}$ which is equal to 1, so also $S(n, n)$.

Thus the total sum is $S(n, 1) + S(n, 2) + \cdots + S(n, n) = B_n$ as desired.

4. $B_n$ counts all ways to partition an $n$ element set. Fix $a$ in the set. Suppose there are $j$ elements that are not in the class with $a$ so $j$ ranges from 0 to $n-1$. The number of such partitions is $\binom{n-1}{j} B_j$ since we must choose the $j$ elements and then
partition them in $B_j$ ways. So $B_n = \sum_{j=0}^{n-1} \binom{n-1}{j} B_j$, as desired.

5. This is just the number of solutions to $C_1 + C_2 + \cdots + C_8 = 15$, which is $\binom{22}{7}$ by Theorem 3.1.

6. The easiest way to do this problem is imagine the 3 shelves side by side. Instead we can think of one long shelf where we must place all the books together with two dividers, which split the long shelf into the 3 shelves. The total number of ways to do this is just the multinomial coefficient:

$$\binom{n + r + s + t + 2}{n, r, s, t, 2}.$$ 

7. The first problem just the Stirling number $S(12, 4)$ by definition. If the bags are now distinct children, each such arrangement has $4!$ labels with the childrens' names so the answer is $4!S(12, 14)$.

8. Using the table of Stirling numbers in the book it’s easy to add up the rows and get that the Bell numbers $B_1$ through $B_8$ are

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147$$

The ratios are 2, 2.5, 3, 3.4, 3.9, 4.3, 4.7, 5.1. So for example it seems that when $n$ gets large, $B_n \geq 5B_{n-1}$ which suggest $B_n$ grows exponentially with $n$, for example bigger than some constant times $5^n$. Much more is known, the Wikipedia article on the Bell numbers is a good place to start.