Review

Level curves of \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) are curves \( f(x,y) = k \), \( k \) constant.

If \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) we get level surfaces \( f(x,y,z) = k \).

- Useful for graphing.

**Ex** \( \text{fixy} = ye^y \)

\[ f(x,y) = x^2y^2 + z^2 \]

Function plots on maple.

**Limits** Compare \( f(x,y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \) and \( g(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \)

as \( (x,y) \rightarrow (0,0) \)

Recall: \( \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \)

**Claim** \( \lim_{(x,y) \rightarrow (0,0)} f(x,y) = 1 \)

**Proof**

Define \( \lambda \) a small disc around \((0,0)\), not including \((0,0)\).

Suppose domain \( D \) of \( f \) contains a small disc around \((0,0)\).

Say \( \lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \) if for every \( \epsilon > 0 \) \( \exists \delta > 0 \) such that if \( (x,y) \in D \) and \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \) then \( |f(x,y) - L| < \epsilon \).

\[ y \]

\[ (a,b) \]

\[ (L-\epsilon, L+\epsilon) \]

Entire \( \delta \) ball around \((a,b)\) maps into \((L-\epsilon, L+\epsilon)\).
In Calc 1, x→a from left or right. In multivariable (x,y) can approach (a,b) along any curve!

\[
\lim_{(x,y) \to (a,b)} \frac{x^3-y^3}{x^2+y^2} \quad \text{DNE}
\]

Pt: On line \( y = ax \),

\[
g_{(x,y)} = \frac{x^3 - a^2 x^3}{x^2 + a^2 x^2} = \frac{(1-a^2)x^3}{(1+a^2)x^2} = \frac{1-a^2}{1+a^2}
\]

or on x axis \( x = 1 \), on y axis \( y = -1 \).

\[
f_{(x,y)} = \frac{xy}{x^2+y^2} \quad \text{on axes } f = 0, \text{ on line } y = x \quad f = \frac{1}{2}.
\]

Limit DNE

\[
f_{(x,y)} = \frac{xy^2}{x^2+y^2}
\]

Does \( \lim_{(x,y) \to (a,b)} f_{(x,y)} \) exist?

* Same limit on any line \( y = mx \)
* Different limits on \( x = y^3 \)

Proving limits exist is difficult!

Recall \( f_{(x,y,z)} \) is continuous at \( (a,b,c) \) if

\[
\lim_{(x,y,z) \to (a,b,c)} f_{(x,y,z)} = f(a,b,c)
\]

Exs: polynomials, exponentials, logs, trig, etc.
\[ \frac{xy}{1+e^{x-y}} \text{ continuous everywhere} \]

\[ \ln(1+x-y) \text{ continuous when } 1+x-y > 0 \]

\[ \frac{e^x + e^y}{e^{xy} - 1} \text{ continuous except if } x = 0 \lor y = 0 \]

14.3 Partial Derivatives

Idea \( f(x,y) \) function of \( 2 \) variables, hold all but one constant.

Ex

\[ f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} \]

Similarly \( f_y(a,b) \)

\[ f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \]

Notation \( z = f(x,y) \) then

\[ f_x(x,y) = f_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1 = D_x f \]
How to calculate? Hold other variables as constant

Ex \( f(x, y) = x^2 + x\sin y + e^{x^2y^3} \)

\( f_x = 2x + \sin y + 2xe^{x^2y^3} \)

\( f_y = x\cos y + 2ye^{x^2y^3} \)

Ex \( x^3 + y^3 + z^3 + 6xyz = 1 \). Find \( \frac{\partial z}{\partial x} \) w/ implicit diff.

\[ 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0 \]

\[ \frac{\partial z}{\partial x} = \frac{x^2 + 6yz}{3z^2 + 3xy} \]

Ex \( f(x, y, z) = e^{xyz^2} \) Find partial deriv.

Ex \( f = x^3 + xy + x^3y^2 \) Find \( \frac{\partial^2 f}{\partial x \partial y} \) and \( \frac{\partial^2 f}{\partial y \partial x} \)

\( f_{xy} \quad f_{yx} \)

Thin (Clarivate) Suppose \( f \) defined on a disc \( D \) containing \( (a, b) \) and \( f_{xy}, f_{yx} \) continuous on \( D \). Prog

Then \( f_{xy}(a, b) = f_{yx}(a, b) \)

= equality of mixed partials
Problems 14.3

#4 Estimate $f_r(40.15)$ from table.

#55 $Z = \frac{y}{2x+3y}$ Find all 2nd partials.

#5-8 w/ visualize

#77 Verify $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of Laplace Eq $U_{xx} + U_{yy} + U_{zz} = 0$