Review

- Parametrized surface $\tilde{r}(uv)$ for $(uv) \in D$.
  - $\tilde{r}_u(u,v)$ and $\tilde{r}_v(u,v)$ are tangent to surface
  - $\tilde{r}_u(u_0,v_0)$, $\tilde{r}_v(u_0,v_0)$ are tangent at $\tilde{r}(u_0,v_0)$

  Tangent plane: $\tilde{r}(st) = r_0(u_0,v_0) + s \tilde{r}_u(u_0,v_0) + t \tilde{r}_v(u_0,v_0)$
  - Normal vector $\hat{n} = \tilde{r}_u(u_0,v_0) \times \tilde{r}_v(u_0,v_0)$

Surface integrals

- $f(x,y,z)$ defined on $S$ param by $\tilde{r}(uv)$ $(uv) \in D$. Then

$$\iint_S f(x,y,z) \, dS = \iint_D f(\tilde{r}(uv)) \left| \tilde{r}_u \times \tilde{r}_v \right| \, dA$$

* $f=1$ gives formula of surface area.

Oriented Surfaces

- Examples
  - $\hat{n} = \frac{\tilde{r}_u \times \tilde{r}_v}{\left| \tilde{r}_u \times \tilde{r}_v \right|}$ unit normal vector.

- Arbitrary choice for closed surfaces:
  - "Positive orientation" = "$\hat{n}$ points out"

Problem

Fluid flow, surface $S$, how much is flowing through $S$?

$F = \text{vector field}$

* Only component of $F$ normal to $S$ contributes to flow through $S$

* This component is $F \cdot \hat{n} = F \cdot \frac{\tilde{r}_u \times \tilde{r}_v}{\left| \tilde{r}_u \times \tilde{r}_v \right|}$
Function is \( \vec{F} : \vec{n} \) so \( \int_S \int_S \vec{F} \cdot \vec{n} \, dS \). Conclude \( \vec{F} \) continuous vector field defined on oriented surface \( S \) w/ normal vector \( \vec{n} \), the surface integral of \( \vec{F} \) over \( S \) is

\[
\int_S \vec{F} \cdot \vec{n} \, dS = \int_D \int_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, ds
\]

\( \text{a.k.a. flux of } \vec{F} \text{ across } S \)

**Example**

\( S \) is helicoid \( \vec{r}(u,v) = (ucosv, usinv, v) \) \( 0 \leq u \leq 1 \) \( 0 \leq v \leq \pi \) w/ upward orientation

\( \vec{F} = (z, y, x) \) Find flux of \( \vec{F} \) across \( S \)

**Answer** \( \vec{r}_u = (cosv, sinv, 0) \) \( \vec{r}_v = (-usinv, ucosv, 1) \) \( \vec{r}_u \times \vec{r}_v = (sinv, -cosv, u) \)

\( \uparrow \text{points up for } 0 \leq u \leq 1 \)

\[
\vec{F}(\vec{r}(u,v)) = (v, usinv, ucosv)
\]

\[
\int_S \vec{F} \cdot \vec{n} = \int_0^1 \int_0^\pi (v, usinv, ucosv) \cdot (sinv, -cosv, u) \, dv \, du
\]

\[
= \frac{1}{2} \int_0^\pi v \sin v - u \sin v \cos v + u^2 \cos v \, dv \, du
\]

\[
= \frac{1}{2} \int_0^\pi v \sin v - u \sin v \cos v + u^2 \cos v \, dv \, du
\]

\[
= \frac{1}{2} \int_0^\pi v \sin v \, dv = \left[ -\cos v \right]_0^\pi = 2
\]
Evaluate \( \iiint F \cdot dS \) \( \text{w/ outward orientation} \)

Six surfaces to parametrize. But can simplify.

\[
\begin{align*}
F &= (x, y, z) \\
F &= (\frac{1}{2}x, y, z) \\
\end{align*}
\]

Ex. \( \hat{u}(x, y, z) \) = temperature. Heat flow is \( \vec{F} = -k \nabla \hat{u} \)

Then \( \iiint \vec{F} \cdot d\vec{S} = -k \iint \nabla \hat{u} \cdot d\vec{S} \) measures rate of heat flow through surface.

Stokes' Thm and Gauss' Thm

Stokes' Thm: \( S \) oriented piecewise smooth surface bounded by simple, closed, piecewise smooth boundary curve \( \partial S \) oriented.

\( \vec{F} \) a vector field \( \text{w/ components having continuous partials} \)

Then \( \iiint \text{curl} \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r} \)

walk around, head pts toward it \( \text{surface on left} \)
Motivation: Work done by $\vec{F}$ around curve = total of circulation.

Special Case: $S$ lies in xy plane so unit normal is $(0,0,1)$, we get

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot (0,0,1) = \iint_S \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

*Green's Thm is a special case of Stokes' Thm.*

16.8.1

Ex: Use Stokes' Thm to evaluate $\iint_S \text{curl} \vec{F} \cdot dS$

$\vec{F} = (x^2 \sin z, y^2, xy)$

$S =$ paraboloid $Z = 1 - x^2 - y^2$ above xy plane oriented up.

16.8.10

Ex: Use Stokes' Thm to evaluate $\oint_C \vec{F} \cdot d\vec{r}$, C ccwwise viewed from above.

$\vec{F} = (2y, xz, x+y)$

$C =$ curve of intersection of $Z = y+3$, cylinder $x^2 + y^2 = 1$

Gauss (Divergence) Thm

E simple solid region in $\mathbb{R}^3$ w/ boundary $S$ oriented positively.

$\vec{F}$ a vector field w/ continuous partials.

Then $\iiint_E \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}$