Lecture 23

C simple, closed ccw

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \]

Green's Thm

Example C boundary enclosed by \( y = x^2, x = y^2, \) ccw

Find

\[ \oint_C (y e^{x^2}) dx + (2x + \cos(y^2)) dy. \]

Remark SS is much easier.

Generalize Suppose region has a hole

Green applies to \( D', D'' \) so

\[ \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_{D'} - \iint_{D''} = \oint_{C''} \oint_{C'} \text{ cancels!} \]

\[ = \oint_{C_1} P \, dx + Q \, dy + \oint_{C_2} P \, dx + Q \, dy \]

* Note \( C_1, C_2 \) have region on left.
Ex. \( F(x,y) = -y i + x j \). Show \( \oint_C \vec{F} \cdot d\vec{r} = 2\pi \) for any simple closed curve about \( \text{origin} \).

**Proof** Notice \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \)

\[
\oint_C \vec{F} \cdot d\vec{r} = \iint_D \nabla \times \vec{F} \, dA = 0
\]

\( C' \): boustrophedon \( C' \) < calculate

but \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \), so why is this not independent of path?

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165 Curl & Divergence

**Notation** Think of \( \nabla \) as operate \( \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \)

**Rank** \( \nabla f \) = \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \) as before.

**Def.** Let \( \vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z)) \) be vector field on \( \mathbb{R}^3 \).

**The curl** \( \text{curl} \vec{F} = \nabla \times \vec{F} = \left( \frac{\partial}{\partial y} R - \frac{\partial}{\partial z} Q, \frac{\partial}{\partial z} P - \frac{\partial}{\partial x} R, \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \times 1PQR \), so

\[
\text{curl} \vec{F} = \begin{vmatrix}
\frac{\partial R}{\partial y} & -\frac{\partial Q}{\partial z} & \frac{\partial P}{\partial x} \\
\frac{\partial R}{\partial z} & \frac{\partial P}{\partial x} & -\frac{\partial P}{\partial y} \\
\frac{\partial R}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z}
\end{vmatrix}
\]
Ex \( \vec{F} = (xy^2z^3, x^2y, xy + z^3) \) Find curl \( \vec{F} \)

\[
\text{curl } \vec{F} = (x - 0, 2xyz - y, 2xy - 2xyz^2)
\]

Then suppose \( f(x,y,z) \) has continuous second order partials. The

\[
\text{curl}(\nabla f) = 0
\]

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (0,0,0) \text{ by Clairaut}
\]

Restate. Conservative vector fields have curl 0.

Rank \( \vec{F} = (P(x,y), Q(x,y), 0) \) then curl \( \vec{F} = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \)

Then suppose \( \vec{F}(x,y,z) \) defined on simply connected region and curl \( \vec{F} = 0 \)

Then \( \vec{F} \) is conservative.

Rank Generalizes 2-dim

Ex \( \vec{F} = (e^{yz}, xe^{yz}, yxe^{yz}) \) is \( \vec{F} \) conservative? If so find a potential \( F(x,y) \).

Question What does curl \( \vec{F} \) measure?

* Points \( \perp \) to rotation, length is rotation.
* \( \text{curl } \vec{F} = 0 \) called irrotational.

Ex \( F(x,y) = [-y, x] \) curl 0?
Def: \( \vec{F} = (P, Q, R) \) the divergence is

\[
\text{div} \ \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

Exercise \( \text{div} (\text{curl} \vec{F}) = 0 \)

Summary
- \( f \) a function, \( \nabla f \) is a vector field
- \( \vec{F} = (P, Q, R) \) a vector field, \( \text{curl} \ \vec{F} \) also a vector field
- \( \vec{F} = (P, Q, R) \) a vector field, \( \text{div} \ \vec{F} \) is a function.

Q: What does divergence measure?

\underline{incompressible}

Problem: Is there a vector field \( \vec{G} \) so

\[
\text{curl} \ \vec{G} = (x \sin y, \cos y, z - xy)
\]

Divergence is \( \sin y - \sin y + z \neq 0 \) so no!

Problem: \( F = (\ln(x^2 + 3z), \ln(x^2 + 3z), \ln(x + 2y)) \)

Find \( \text{curl} F \) & \( \text{div} F \).