<table>
<thead>
<tr>
<th>Problem #</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
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<tr>
<td>3</td>
<td>10</td>
<td></td>
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<tr>
<td>4</td>
<td>10</td>
<td></td>
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<tr>
<td>5</td>
<td>15</td>
<td></td>
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<tr>
<td>6</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>100</strong></td>
<td></td>
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</tbody>
</table>
Instructions: You may not use any outside help, including notes, index cards, electronic devices or your neighbor!

1. (25 points) Short answer, very limited partial credit.

   a. Define: A vector field $\mathbf{F}$ is conservative if $\cdots$

   $$\mathbf{F} = \nabla f$$ for some function $f$.

   b. Sketch a region in the plane which is connected but is not simply connected.

   ![Region Sketch]

   c. Let $\mathbf{F}(x, y, z) = (y + 2xz, x + z, y + x^2)$. Find a potential function for $\mathbf{F}$.

   

   $$f_x = y + 2xz \text{ so } f = xy + x^2z + (y, z)$$

   $$f_y = x + z \text{ so } f = xy + yz + (y, z)$$

   $$f_z = y + x^2 \text{ so } f = x^2y + z + (y, z, 1)$$

   $$f(x, y, z) = xy + yz + x^2z$$
d. Define: A vector field $\mathbf{F}$ is irrotational if \[
\text{curl} \mathbf{F} = \vec{0}
\]

e. State the Fundamental Theorem for Line Integrals.

Let $C$ be a smooth curve given by $\hat{r}(t)$ for $a \leq t \leq b$. Suppose $\nabla f$ is continuous on $C$. Then
\[
\oint_C \nabla f \cdot d\hat{r} = f(\hat{r}(b)) - f(\hat{r}(a))
\]

f. Sketch a vector field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ on $\mathbb{R}^2$ such that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path.

Any field w/ obvious rotation.
2. (10 points) Consider the surface parameterized by \( r(s, t) = (t, s^2t, t^2 + 3s) \). Find the equation of the tangent plane to this surface at the point \((2, 2, 7)\). Give your answer in two forms. First parameterize the equation of the plane. Then put your answer in the form \( ax + by + cz = d \).

\((2, 2, 7)\) is \( s = 1 \) \( t = 2 \)

\( r_s = (0, 2st, 3) \) \( \quad r_s(1, 2) = (0, 4, 3) \)

\( r_t = (1, s^2, 2t) \) \( \quad r_t(1, 2) = (1, 4, 4) \)

\( r_s \times r_t = (13, 3, -4) \)

\((2, 2, 7) + s(0, 4, 3) + t(1, 4, 4) \)

\( 13x + 3y - 4z = 60 \)
3. (10 points) Let \( \mathbf{F}(x,y,z) = (xy^2, z \cos(y), x + y + z) \). Calculate the curl and the divergence of \( \mathbf{F} \).

\[
\text{curl } \mathbf{F} = (1 - \cos y, -1, -2xy)
\]

\[
\text{div } \mathbf{F} = y^2 - z \sin y + 1
\]

4. (10 points) Let \( C \) be the curve \( r(t) = (t, t^2, t^3) \) and let \( \mathbf{F}(x,y,z) = (xy, x - y, z) \). Calculate the work done by \( \mathbf{F} \) moving a particle along the curve from \( (0,0,0) \) to \( (2,4,8) \).

\[
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \mathbf{F}(r(t)) \cdot \frac{d\mathbf{r}}{dt} \, dt
\]

\[
= \int_0^2 (t^3, t - t^3, t^3) \cdot (1, 2t, 3t^2) \, dt
\]

\[
= \int_0^2 t^3 + 2t^4 - 2t^4 + 3t^5 \, dt
\]

\[
= \int_0^2 t^3 + 3t^5 \, dt
\]

\[
= \left[ \frac{1}{4} t^4 + \frac{3}{2} t^6 \right]_0^2
\]

\[
= \frac{1}{4} (2)^4 + \frac{3}{2} (2)^6
\]

\[
= 32 - 4 + \frac{16}{3}
\]

\[
= 28 + \frac{16}{3} = \frac{100}{3}
\]
5. (15 points) Use the Divergence Theorem to calculate the surface integral \( \iint_S \mathbf{F} \cdot d\mathbf{S} \); that is, calculate the flux of \( \mathbf{F} \) across \( S \).

\[
\mathbf{F}(x, y, z) = (2x^3 + y^3)i + (y^3 + z^3)j + 3y^2zk
\]

and \( S \) is the surface of the solid bounded by the paraboloid \( z = 1 - x^2 - y^2 \) and the \( xy \)-plane.

\[
\nabla \cdot \mathbf{F} = 6x^2 + 3y^2 + 3z^2 = 6x^2 + 6y^2
\]

Want \( \iiint_E 6x^2 + 6y^2 \, dV \). Use cylindrical.

\[
0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1, \quad 0 \leq z \leq 1 - r^2
\]

\[
\int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6r^3 \, dz \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 6r^3(1-r^2) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left[ \frac{3}{2} r^4 - r^6 \right]_0^1 \, d\theta
\]

\[
= \frac{\pi}{2}
\]
6. **(15 points)** Let $C$ be the boundary of the half disk $x^2 + y^2 \leq a^2$, $y \geq 0$, oriented counterclockwise. Use Green's Theorem to evaluate:

\[
\oint_C (\sin x + 3y^2) \, dx + (2x - e^{-y^2}) \, dy.
\]

\[\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 2 - 6y \text{ so we want } \iint_D 52 - 6y \, dA.\]

Using polar: $0 \leq \theta \leq \pi$, $0 \leq r \leq a$

\[
\int_0^a \int_0^a (2-6r \sin \theta) \, r \, dr \, d\theta = \int_0^a 2r - 6r^2 \sin \theta \, dr \, d\theta
\]

\[= \int_0^a r^2 - 2r^3 \sin \theta \, d\theta \]

\[= \left[ \frac{r^3}{3} - \frac{2r^4}{4} \sin \theta \right]_0^a \]

\[= \frac{a^3}{3} - \frac{2a^4}{4} \sin \theta \bigg|_0^a \]

\[= 2\pi a^2 + 2a^3 - 2\pi a^3 - 2a^3 \]

\[= 2\pi a^2 + 4a^3 \]

\[= (\pi a^2 - 2a^3) - (0 + 2a^3) \]

\[= \sqrt{\pi a^2 + 4a^3} \]

\[\sqrt{\pi a^2 - 4a^3} \]
7. (15 points) Let $S$ be the helicoid parameterized by $r(u, v) = (u \cos v, u \sin v, v)$, $0 \leq u \leq 1$, $0 \leq v \leq \pi/2$.

a. Set up but do not evaluate an integral which gives the surface area of $S$.

b. Let $F(x, y, z) = (z, y, x)$. Evaluate the surface integral $\iint_S F \cdot dS$.

\[ \begin{align*}
ru &= (\cos v, \sin v, 0) & rv &= (-u \sin v, u \cos v, 1) & ru \times rv &= (\sin v, -\cos v, u) \\
|ru \times rv| &= \sqrt{1+u^2}
\end{align*} \]

a. Surface area $\leq \int_0^1 \int_0^{\pi/2} \sqrt{1+u^2} \, dv \, du$

b. $\int_0^1 \int_0^{\pi/2} F(r(u, v)) \cdot (ru \times rv) \, dv \, du = \int_0^1 \int_0^{\pi/2} (v, u \sin v, u \cos v) \cdot (\sin v, -\cos v, u) \, dv \, du$

\[ \begin{align*}
&= \int_0^1 \int_0^{\pi/2} v \sin v - u \sin v \cos v + u^2 \cos v \, dv \, du \\
&= \int_0^1 \int_0^{\pi/2} \left( v \sin v - u \sin v \cos v + u^2 \cos v \right) \, dv \, du \\
&= \int_0^1 \int_0^{\pi/2} \left( v \sin v - \frac{u^2}{2} \sin v + u^2 \sin v \right) \, dv \, du \\
&= \int_0^1 \left( 0 + 1 - \frac{u^2}{2} + u^2 \right) \, du \\
&= \int_0^1 (1 - \frac{u^2}{2} + u^2) \, du = \left[ u - \frac{u^3}{6} + \frac{u^3}{3} \right]_0^1 = 1 - \frac{1}{12} + \frac{1}{3} = \frac{13}{12}
\]