1. **(10 points)** Let \( f(x, y) = x^2 + 2y^2 \). Sketch level curves \( f(x, y) = k \) for \( k = 1, 2, 3, 4 \) and on the same picture sketch the vector field \( \nabla f \).

Recall \( \nabla f \) is \( \perp \) to level curves

b. Find the direction and the rate of maximum increase for \( f(x, y) \) at the point \((1, 3)\).

\[
\nabla f = (2x, 4y) \\
\nabla f(1,3) = (2,12) = \text{Direction} \\
\sqrt{2^2 + 12^2} = \text{Rate}
\]
2. (15 points) Let $S$ be the surface $z = x^2 + y^2$ lying below the plane $z = 2$.

a. Neatly sketch the surface.

b. Parameterize the surface as $\vec{r}(u,v)$ with $(u,v) \in D$. Be sure to describe the region $D$.

c. Let $\vec{F}(x,y,z) = (z,x,y)$. Use Stokes’ theorem to evaluate:

\[ \int \int_S \text{curl} \vec{F} \cdot \vec{n} \, dS. \]

\[ \vec{r}(u,v) = (u, v, u^2 + v^2), \quad (u,v) \in D. \]

Boundary curve $C$: \( \vec{r}(t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 2), \quad 0 \leq t \leq 2\pi \)

\( \vec{r}'(t) = (-\sqrt{2} \sin t, \sqrt{2} \cos t, 0) \)

\[ \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} (\sqrt{2} \cos t, \sqrt{2} \sin t, 0) \cdot (-\sqrt{2} \sin t, \sqrt{2} \cos t, 0) \, dt \]

\[ = \int_0^{2\pi} (-2 \sqrt{2} \sin t \cos t + 2 \cos^2 t) \, dt \]

\[ = \int_0^{2\pi} (-2 \sqrt{2} \sin t \cos t) \, dt + \frac{1}{2} \left[ \int_0^{2\pi} \cos 2t \, dt \right] \]

\[ = -2 \sqrt{2} \int_0^{2\pi} \sin t \cos t \, dt + \frac{1}{2} \left[ \int_0^{2\pi} \cos 2t \, dt \right] \]

\[ = -2 \sqrt{2} \left[ \frac{1}{2} \sin t \cos t \right]_0^{2\pi} + \frac{1}{2} \left[ \frac{\sin 2t}{2} \right]_0^{2\pi} \]

\[ = -2 \sqrt{2} (0) + \frac{1}{2} \left[ \frac{\sin 2t}{2} \right]_0^{2\pi} \]

\[ = \pi \]

= \boxed{2\pi}
3. (15 points) Let $C$ be the curve of intersection of the plane $y + z = 7$ and the cylinder $x^2 + y^2 = 4$, oriented to be counterclockwise when viewed from above. Use Stokes' theorem to compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = (-y^2, x, z^2)$.

Parametrize $S$ as

$$\vec{r}(u, v) = (u, v, 7 - v) \quad (u, v) \epsilon \mathbb{R}^2$$

Stokes

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \hat{n} \, dA$$

$$\vec{r}_u = (1, 0, 0) \quad \vec{r}_v = (0, 1, -1) \quad \vec{r}_u \times \vec{r}_v = (0, 1, 1) \quad \text{points up!}$$

$$\text{curl} \vec{F} = (0, 0, 1 + 2y)$$

$$\iint_D (0, 1 + 2v) \, dA = \iint (1 + 2v) \, dA$$

Let $u = r \cos \theta, v = r \sin \theta$

$$= \int_0^{2\pi} \int_0^2 (1 + 2r \sin \theta) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[ \frac{r^2}{2} (1 + 2r \sin \theta) \right]_0^2 \, d\theta$$

$$= \int_0^{2\pi} (2r^2 \sin \theta - 2r^2 \cos \theta) \, d\theta$$

$$= \int_0^{2\pi} 2r^2 \, dr \int_0^1 \cos \theta \, d\theta$$

$$= \frac{1}{2} \pi r^2 = \pi r^2 \int_0^1$$

$$= \frac{1}{2} \pi r^2$$
4. (25 points) Let $S$ be the cylinder (including top and bottom) $x^2 + y^2 = 1, 0 \leq z \leq 1,$ with outward pointing normal. Let $\vec{F}(x, y, z) = (x^3, y^3, e^{-z}).$ Use the divergence theorem to compute

$$\int \int_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_E \text{div} \vec{F} \, dV$$

$E: 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq 1$ in cylindrical coordinates

$$\text{div} \vec{F} = 3x^2 + 3y^2 - e^{-z} = 3r^2 - e^{-z}$$

$$\int_0^{2\pi} \int_0^1 \int_0^1 (3r^2 - e^{-z}) \, r \, dz \, d\theta \, dr$$

$$= \int_0^{2\pi} \int_0^1 3r^3 z + re^{-z} \bigg|_{z=0}^1 \, d\theta \, dr$$

$$= \int_0^{2\pi} \int_0^1 3r^3 + \frac{1}{e} r - r \, d\theta \, dr$$

$$= \int_0^{2\pi} \left( 3r^3 + \frac{1}{e} r - r \right) \, dr$$

$$= 2\pi \left( \frac{3}{4} r^4 + \frac{1}{2e} r^2 - \frac{r^2}{2} \right) \bigg|_0^1$$

$$= 2\pi \left( \frac{3}{4} + \frac{1}{2e} - \frac{1}{2} \right)$$

$$= 2\pi \left( \frac{3}{4} + \frac{1}{2e} - \frac{1}{2} \right)$$

$$= \frac{\pi}{2} + \pi \sqrt{\frac{1}{e}}$$
5. (10 points) Let $S$ be the portion of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 1$ and $z = 4$. Write down but do not evaluate an integral which gives the surface area of $S$.

\[ S: \quad \vec{r}(u,v) = (u, v, u^2 + v^2) \]

\[ (u, v) \in \mathbb{D} \]

\[ ru = (1, 0, 2u) \quad rv = (0, 1, 2v) \]

\[ ru \times rv = (-2u, -2v, 1) \]

\[ |ru \times rv| = \sqrt{1 + 4u^2 + 4v^2} \]

\[ A(S) = \iint_D \sqrt{1 + 4u^2 + 4v^2} \, dA \]

\[ = \int_0^2 \int_1^2 \sqrt{1 + 4u^2} \, r \, dQ \, dr \]
6. (10 points) Consider the surface parameterized by \( \mathbf{r}(u,v) = (u^2 + v^2, uv, uv^2) \). Find the tangent plane to this surface at the point where \( u = 1, v = 2 \). Give the equation of the tangent plane in two forms, first parameterize it. Second, give it in \( ax + by + cz = d \) form.

\[
\hat{\mathbf{r}}_y = (2u, v, v^2) \quad \hat{\mathbf{r}}_v = (2v, u, 2uv)
\]

\[
A + I = u, 2v \quad \hat{\mathbf{r}}_y = (2, 2, 4) \quad \hat{\mathbf{r}}_v = (4, 1, 4)
\]

\[
\mathbf{r}_y \times \mathbf{r}_v = (4, 8, -6) \quad \text{use as } \mathbf{n} \quad \text{point is } (5, 2, 4)
\]

\[
4x + 8y - 6z = 12
\]

\[
\mathbf{r}(s,t) = (5, 2, 4) + s(2, 2, 4) + t(4, 1, 4) \quad -\infty < s, t < \infty
\]
7. (10 points) Use Green's theorem to calculate the work done by the force field \( \mathbf{F}(x, y) = (x^2 - y^3, x^3 + y^2) \) moving a particle around the unit circle \( x^2 + y^2 = 1 \) in the clockwise direction.

Green: \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

\[ = \iint_D 3x^2 + 3y^2 \, dA \]

\[ = \int_0^{2\pi} \int_0^1 3r^2 \, r \, dr \, d\theta \]

\[ = \int_0^{2\pi} \frac{3}{4} r^4 \bigg|_0^1 = 2\pi \frac{3}{4} = \frac{3\pi}{2} \]

but we want clockwise so

\[ -\frac{3\pi}{2} \]
8. (10 points) Let \( f(x, y, z) = x^2y + y^2z + xyz \).

   a. Calculate \( \nabla f \) and \( \text{curl}(\nabla f) \).

   b. Calculate the line integral: \( \int_C \nabla f \cdot d\vec{r} \) where \( C \) is the top half of the ellipse \( 3x^2 + 7y^2 = 12 \) traversed from \((-2, 0)\) to \((2, 0)\) in \( xy \) plane.

   \[ \nabla f = (2xy + yz, x^2 + yz + xz, xy + yz) \]

   \[ \text{curl}(\nabla f) = (0, 0, 0) \]

   \[ f(2, 0, 0) - f(-2, 0, 0) = 0 - 0 = 0 \]
9. (10 points) Let \( f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16 \). Find and classify all critical points of \( f(x, y) \) as local max, local min or saddle points.

\[ \nabla f = (4x^3 - 16x, 2y - 6) \]

\[ 4x^3 - 16x = 0 \quad 2y - 6 = 0 \]

\[ x = 0, 2, -2 \quad y = 3 \]

Critical Points:

- \((0, 3)\)
- \((2, 3)\)
- \((-2, 3)\)

Calculating the discriminant:

\[ D = f_{xx} f_{yy} - f_{xy}^2 = (12x^2 - 16)2 - 0 = 24x^2 - 32 \]

- For \((0, 3)\):
  \[ D = -32 \] saddle

- For \((2, 3)\):
  \[ D = 64 \] \( f_{xx} > 0 \) local min

- For \((-2, 3)\):
  \[ D = 64 \] \( f_{xx} > 0 \) local min
10. (10 points) Suppose $w = x^2 e^{yz}$ and $(x, y, z) = (st + u^2, s^2 ut, s + tu)$. Find $\frac{\partial w}{\partial u}$ at the point where $s = 1, t = 2, u = 3$.

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}
\]

\[
= 2x e^{yz} \cdot 2u + x^2 z e^{yz} \cdot s^2 t + yx^2 e^{yz} \cdot 5
\]

\[
x = 11, \ y = 6, \ z = 7
\]

\[
= 22e^{6} \cdot 6 + 121.7 \cdot e^{6} \cdot 2 + 121.6 \cdot e^{6} \cdot 2
\]

\[
= e^{6} (132 + 121.14 + 121.12) = \boxed{3278e^{6}}
\]

11. (10 points) Find the equation of the plane containing the points $(1, 2, 3), (-2, 1, 1), (-3, 1, 2)$.

\[
\vec{PQ} = (-3-1, -2) \quad \vec{PR} = (-1, -1, -1)
\]

\[
\vec{PQ} \times \vec{PR} = (-1, 5, -1)
\]

\[
-x + 5y - z = 6
\]
12. (10 points) Let $D$ be the region enclosed by the curves $y = 0$, $y = x^2$ and $x = 1$. Find the average value of $f(x, y) = x \sin y$ over the region $D$.

\[
\text{Area } D = \iiint_D 1 = \int_0^1 \int_0^{x^2} 1 \, dy \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}
\]

\[
\iiint_D x \sin y \, dy \, dx = \int_0^1 \int_0^{x^2} x \sin y \, dy \, dx = \int_0^1 -x \cos y \bigg|_{y=0}^{y=x^2} \, dx = \int_0^1 -x \cos (x^2) + x \, dx = \left[ -\frac{1}{2} \sin (x^2) + \frac{x^2}{2} \right]_0^1 = -\frac{1}{2} \sin (1) + \frac{1}{2}
\]

\[
\text{Fave} = \frac{\iiint_D f(x, y)}{\text{Area}} = \frac{-\frac{1}{2} \sin (1) + \frac{1}{2}}{\frac{1}{3}} = -\frac{3}{2} \sin (1) + 3/2
\]
13. (5 points) Consider a particle with position at time $t$ given by $\vec{r}(t) = (1 + t, t^3, t^2)$. What is the speed of the particle at when $t = 3$?

\[ \vec{r}' = (1, 3t^2, 2t) \]
\[ \vec{r}'(3) = (1, 27, 6) \]
\[ |\vec{r}'(3)| = \sqrt{1^2 + 27^2 + 36} = \sqrt{764} \]

14. (10 points) Let $R$ be the rectangle $[-3, 3] \times [0, 3]$. Estimate $\int_R xy^2$ using a Riemann sum with $m = n = 3$ and the upper right corner of each rectangle as your sample point.

Each rect has area $2$.

\[ 2 \left( f(-1, 1) + f(1, 1) + f(1, 3) + f(-1, 3) + f(1, 2) + f(1, 3) \right. \]
\[ + f(-1, 3) + f(1, 3) + f(1, 3) \left. \right) \]
\[ = 2 \left( -1 + 1 + 3 - 4 + 4 + 12 + 9 + 9 + 27 \right) \]
\[ = 2 (42) = 84 \]