Name: SOLUTIONS

Math 2950- Midterm Exam #1 - September 27, 2004

1. **(10 points)** Find the equation of the plane perpendicular to the line

$$(x, y, z) = (1, -2, 3) + t(1, -2, 1)$$

and passing through the point $(2, 4, -1)$.

We know the normal vector is $(1,-2,1)$

$$\boxed{\langle x-2, y-4, z+1\rangle \cdot \langle 1,-2,1\rangle = 0}$$

or expanded out

$$\boxed{x-2y+z = -7}$$

2. **(15 points)** Let $\vec{u} = (1, -1, 3)$, $\vec{v} = (1, 1, 2)$. Calculate $\vec{u} \cdot \vec{v}$ and $\vec{u} \times \vec{v}$. Then find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = 1-1+6 = \boxed{6}$$

$$\vec{u} \times \vec{v} = (-2-3, -2+3, 1-(-1)) = (-5, 1, 2)$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{6}{\sqrt{11}\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{66}}\right)$$

3. **(10 points)**. Let $\vec{u}$ be a differentiable vector funtion and f a real valued function. Then the chain rule says:

$$\frac{d}{dt}[\vec{u}(f(t))] = ?$$

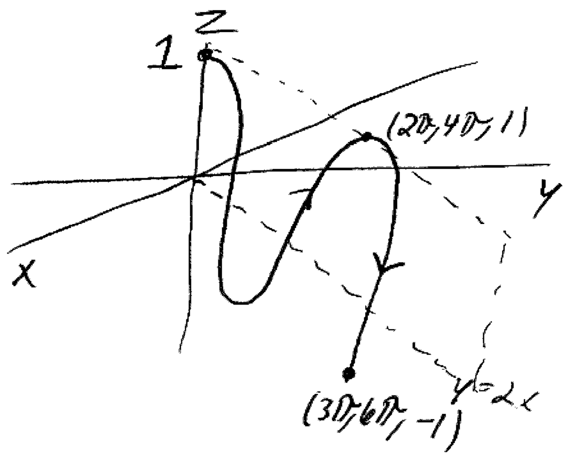
$$\vec{u}'(f(t))f'(t)$$

4. **(10 points)** Sketch the space curve

$$\vec{r}(t) = (t, 2t, \cos(t))$$

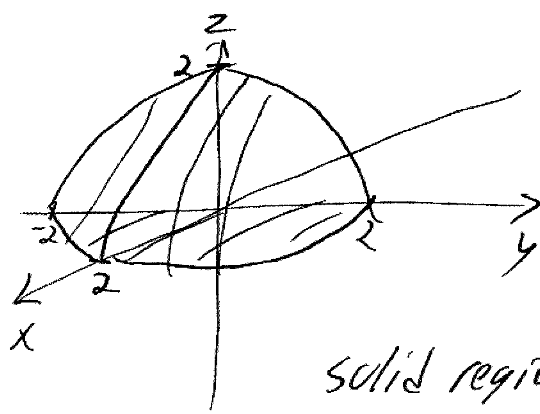
for the interval $0 \leq t \leq 3\pi$, indicating with an arrow the direction of increasing t.

Notice the curve lies on the plane $y = 2x$.



5. **(10 points)** Sketch the region given by the spherical coordinate inequalities:

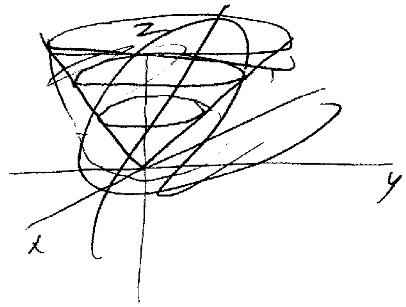
$$0 \leq \phi \leq \pi/2$$
$$-\pi/2 \leq \theta \leq \pi/2$$
$$0 \leq \rho \leq 2.$$

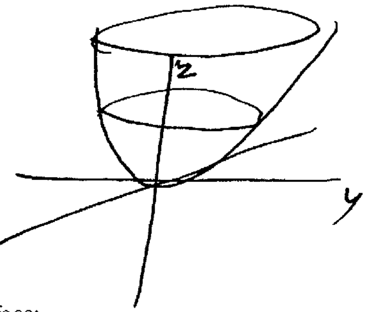


solid region, 1/2 of the upper hemisphere.

6. **(20 points)**

a. Sketch the surface given by $z = x^2 + y^2$.





b. Verify that the two space curves below both lie on the surface:

$$r(\vec{t}) = (\cos(t), \sin(t), 1) \text{ and } \vec{x}(s) = (s, 0, s^2)$$

$1 = \cos^2 t + \sin^2 t$ so $\vec{r}(t)$ is on the surface

$s^2 = s^2 + 0^2$ so $\vec{x}(s)$ is on the surface

c. Verify that the point $(1, 0, 1)$ lies on both curves. (i.e. find a t_0 such that $(1, 0, 1) = \vec{r}(t_0)$ and a s_0 such that $(1, 0, 1) = \vec{x}(s_0)$.)

$(1,0,1) = \vec{r}(0)$ so $(1,0,1)$ is on $\vec{r}(t)$

$(1,0,1) = \vec{x}(1)$ so $(1,0,1)$ is on $\vec{x}(s)$

d. Find the equation for the tangent line to the curve $\vec{r}(t)$ at the point $(1,0,1)$.

$$(x,y,z) = (1,0,1) + t\,(0,1,0)$$

$r'(t) = (-\sin t, \cos t, 0)$

$r'(0) = (0,1,0)$

e. Find the equation for the tangent line to the curve $\vec{x}(s)$ at the point $(1,0,1)$.

$\vec{x}'(s) = (1,0,2s)$

$\vec{x}'(1) = (1,0,2)$

$(x,y,z) = (1,0,1) + t\,(1,0,2)$

f. Find the equation of the plane passing through the point $(1,0,1)$ and containing both tangent lines above. This is called the *tangent plane* to the surface at the point $(1,0,1)$.

The normal vector is $\perp$ to both direction vectors for the lines. Thus

$$\vec{n} = (1,0,2) \times (0,1,0) = (-2,0,1)$$

$$(x-1, y, z-1)\cdot(-2,0,1) = 0$$

or

$$\boxed{-2x + z = -1}$$

7. **(15 points)** a. Find the velocity, acceleration, and speed of a particle with position function given by:

$$r(t) = (t^2 + 1, \cos(t), t)$$

$\vec{v}(t) = (2t, -\sin t, 1)$

$\vec{a}(t) = (2, -\cos t, 0)$

speed $= |\vec{v}(t)| = \sqrt{4t^2 + \sin^2 t + 1}$

b. Express the arc length of the curve above from $t = 1$ to $t = 2$ as a definite integral (do not try to evaluate the integral!).

$\int_1^2 \sqrt{4t^2 + \sin^2 t + 1}\, dt$

8. **(10 points)** Find the work by a force $\vec{F} = (1, -1, 1)$ moving an object from the point $(1, 1, 1)$ to the point $(4, 2, 1)$.

$\vec{D} = (3, 1, 0)$

$W = \vec{F} \cdot \vec{D} = 3 - 1 + 0 = 2$