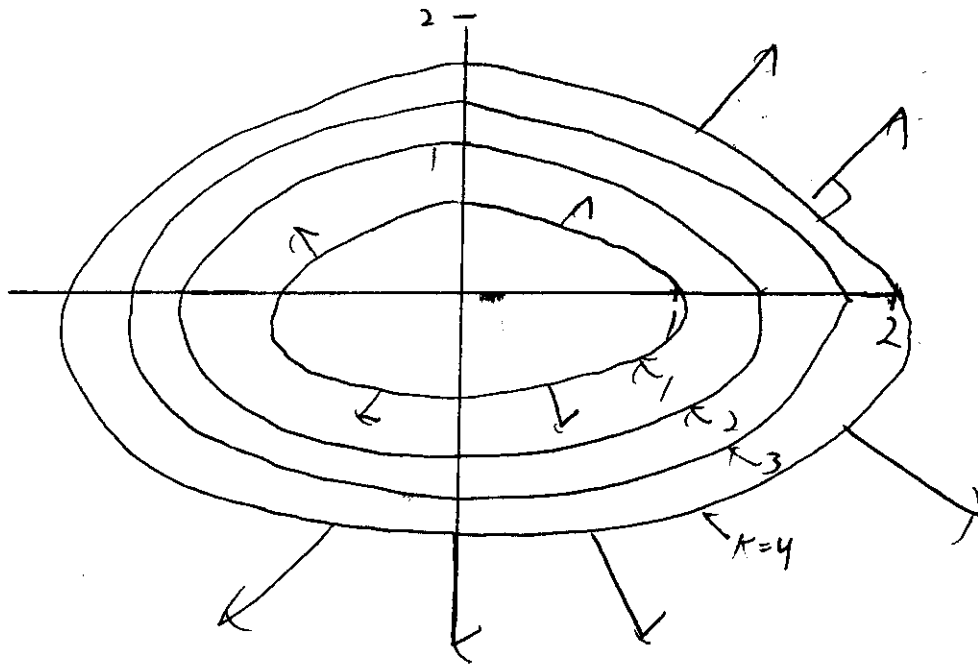


Name: SOLUTIONS

Math 241WW- Final Exam - December 12, 2008

Instructions: The exam is worth 150 points. Calculators are not permitted. You are permitted two index cards, each no larger than 3×5 , with whatever you like written on them.

1. (10 points) Let $f(x, y) = x^2 + 2y^2$. Sketch level curves $f(x, y) = k$ for $k = 1, 2, 3, 4$ and on the same picture sketch the vector field ∇f .



Recall ∇f is \perp to level curves

b. Find the direction and the rate of maximum increase for $f(x, y)$ at the point $(1, 3)$.

$$\nabla f = (2x, 4y)$$

$$\nabla f(1, 3) = (2, 12) \leftarrow \text{Direction}$$

$$\sqrt{2^2 + 12^2} \leftarrow \text{rate}$$

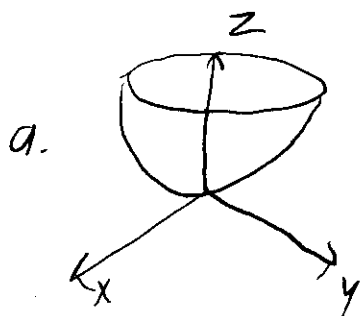
2. (15 points) Let S be the surface $z = x^2 + y^2$ lying below the plane $z = 2$.

a. Neatly sketch the surface.

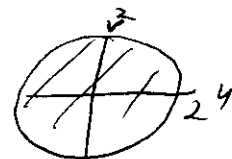
b. Parameterize the surface as $\vec{r}(u, v)$ with $(u, v) \in D$. Be sure to describe the region D .

c. Let $\vec{F}(x, y, z) = (z, x, y)$. Use Stokes' theorem to evaluate:

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS.$$



b. $\vec{r}(u, v) = (u, v, u^2 + v^2) \quad (u, v) \in D$



c. Boundary curve C : $\vec{r}(t) = (\sqrt{2} \cos t, \sqrt{2} \sin t, 2) \quad 0 \leq t \leq 2\pi$
 $\vec{r}'(t) = (-\sqrt{2} \sin t, \sqrt{2} \cos t, 0)$

$$\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = \int_0^{2\pi} (2, \sqrt{2} \cos t, \sqrt{2} \sin t) \cdot (-\sqrt{2} \sin t, \sqrt{2} \cos t, 0) \, dt$$

$$= \int_0^{2\pi} -2\sqrt{2} \sin t + 2 \cos^2 t \, dt$$

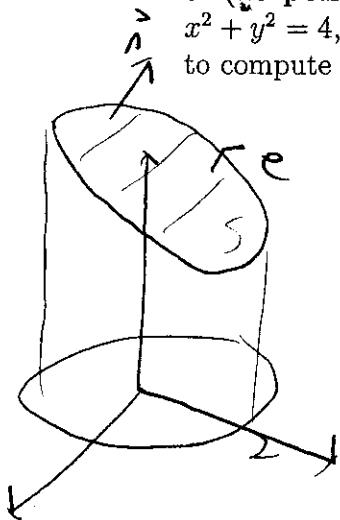
$$= \int_0^{2\pi} -2\sqrt{2} \sin t \, dt + 2 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt$$

$$\parallel \int_0^{2\pi} + 2 \left(\frac{t}{2} + \frac{1}{4} \sin 2t \right) \Big|_0^{2\pi}$$

$$= 2 \cdot \pi$$

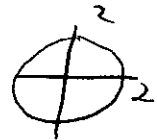
$$= \boxed{2\pi}$$

3. (15 points) Let C be the curve of intersection of the plane $y + z = 7$ and the cylinder $x^2 + y^2 = 4$, oriented to be counterclockwise when viewed from above. Use Stokes' theorem to compute $\int_C \vec{F} \cdot d\vec{r}$ where $F(x, y, z) = (-y^2, x, z^2)$.



parametrize S as:

$$\vec{r}(u, v) = (u, v, 7-v) \quad (u, v) \in D$$



Stokes: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n}$

$$\vec{r}_u = (1, 0, 0) \quad \vec{r}_v = (0, 1, -1) \quad \vec{r}_u \times \vec{r}_v = (0, 1, 1) \text{ points up!}$$

$$\text{curl } \vec{F} = (0, 0, 1+2y)$$

$$\iint_D (0, 0, 1+2v) \cdot (0, 1, 1) = \iint_D (1+2v) \quad \text{Let } u = r \cos \theta \quad v = r \sin \theta$$

$$= \int_0^2 \int_0^{2\pi} (1+r \sin \theta) r \, dr \, d\theta$$

$$= \int_0^2 r \theta - 2r^2 \cos \theta \Big|_0^{2\pi} \, dr$$

$$= \int_0^2 2\pi r \, dr = \pi r^2 \Big|_0^2 = \boxed{4\pi}$$

4. (15 points) Let S be the cylinder (including top and bottom) $x^2 + y^2 = 1, 0 \leq z \leq 1$, with outward pointing normal. Let $\vec{F}(x, y, z) = (x^3, y^3, e^{-z})$. Use the divergence theorem to compute

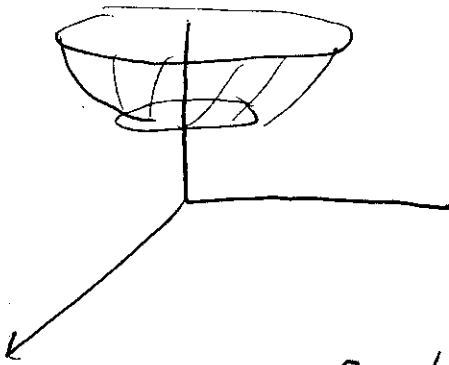
$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_E \operatorname{div} F$$

$E: 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 1$ in cyl coord

$$\operatorname{div} F = 3x^2 + 3y^2 - e^{-z} = 3r^2 - e^{-z}$$

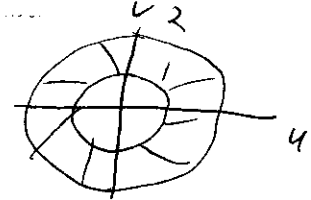
$$\begin{aligned} & \int_0^1 \int_0^{2\pi} \int_0^1 (3r^2 - e^{-z}) r \, dz \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} \left. 3r^3 z + r e^{-z} \right|_{z=0}^1 d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} \left(3r^3 + \frac{1}{e} r - r \right) d\theta \, dr \\ &= \int_0^1 2\pi \left(3r^3 + \frac{1}{e} r - r \right) dr \\ &= 2\pi \cdot \left. \left(\frac{3}{4} r^4 + \frac{1}{2e} r^2 - \frac{r^2}{2} \right) \right|_0^1 \\ &= 2\pi \left(\frac{3}{4} + \frac{1}{2e} - \frac{1}{2} \right) \\ &= 2\pi \left(\frac{1}{4} + \frac{1}{2e} \right) \\ &= \frac{\pi}{2} + \pi/e \end{aligned}$$

5. (10 points) Let S be the portion of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 1$ and $z = 4$. Write down *but do not evaluate* an integral which gives the surface area of S .



$$S: \vec{r}(u,v) = (u, v, u^2 + v^2)$$

$$(u,v) \in$$



$$r_u = (1, 0, 2u) \quad r_v = (0, 1, 2v)$$

$$r_u \times r_v = (-2u, -2v, 1)$$

$$|r_u \times r_v| = \sqrt{1 + 4u^2 + 4v^2}$$

$$A(S) = \iint_D \sqrt{1 + 4u^2 + 4v^2}$$

$$= \int_1^2 \int_0^{2\pi} \sqrt{1 + 4r^2} r \, d\theta \, dr$$

6. (10 points) Consider the surface parameterized by $\vec{r}(u, v) = (u^2 + v^2, uv, uv^2)$. Find the tangent plane to this surface at the point where $u = 1, v = 2$. Give the equation of the tangent plane in two forms, first parameterize it. Second, give it in $ax + by + cz = d$ form.

$$\vec{r}_u = (2u, v, v^2) \quad \vec{r}_v = (2v, u, 2uv)$$

$$u=1, v=2 \quad \vec{r}_u = (2, 2, 4) \quad \vec{r}_v = (4, 1, 4)$$

$$\vec{r}_u \times \vec{r}_v = (4, 8, -6) \text{ use as } \vec{n} \quad \text{point is } (5, 2, 4)$$

$$4x + 8y - 6z = 12$$

$$\vec{r}(s, t) = (5, 2, 4) + s(2, 2, 4) + t(4, 1, 4) \quad -\infty < s < \infty$$

7. (10 points) Use Green's theorem to calculate the work done by the force field $\vec{F}(x, y) = (x^2 - y^3, x^3 + y^2)$ moving a particle around the unit circle $x^2 + y^2 = 1$ in the clockwise direction.



$$\begin{aligned} \text{Green: } \oint_C \vec{F} &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ &= \iint_D 3x^2 + 3y^2 \\ &= \int_0^{2\pi} \int_0^1 3r^2 r dr d\theta \\ &= \int_0^{2\pi} \frac{3}{4} r^4 \Big|_0^1 = 2\pi \cdot \frac{3}{4} = 3\pi/2 \end{aligned}$$

but we want clockwise so

$$\boxed{-3\pi/2}$$

8. (10 points) Let $f(x, y, z) = x^2y + y^2z + xyz$.

a. Calculate ∇f and $\text{curl}(\nabla f)$.

b. Calculate the line integral: $\int_C \nabla f \cdot d\vec{r}$ where C is the top half of the ellipse $3x^2 + 7y^2 = 12$ traversed from $(-2, 0, 0)$ to $(2, 0, 0)$ in xy plane.

$$a. \nabla f = (2xy + yz, x^2 + 2yz + xz, y^2 + xy)$$

$$\text{curl}(\nabla f) = (0, 0, 0)$$

$$b = f(2, 0, 0) - f(-2, 0, 0) = 0 - 0 = 0$$

9. (10 points) Let $f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16$. Find and classify all critical points of $f(x, y)$ as local max, local min or saddle points.

$$\nabla f = (4x^3 - 16x, 2y - 6)$$

$$4x^3 - 16x = 0 \quad 2y - 6 = 0$$

$$x = 0, 2, -2$$

$$y = 3$$

$$\text{C.P. } (0, 3)$$

$$(2, 3)$$

$$(-2, 3)$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 16)2 - 0 = 24x^2 - 32$$

$$(0, 3) \quad D = -32 \quad \text{saddle}$$

$$(2, 3) \quad D = 64 \quad f_{xx} > 0 \quad \text{local min}$$

$$(-2, 3) \quad \text{"} \quad \text{"} \quad \text{local min}$$

10. (10 points) Suppose $w = x^2 e^{yz}$ and $(x, y, z) = (st + u^2, s^2 ut, s + tu)$. Find $\frac{\partial w}{\partial u}$ at the point where $s = 1, t = 2, u = 3$.

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\ &= 2x e^{yz} \cdot 2u + x^2 z e^{yz} s^2 t + y x^2 e^{yz} \cdot t \\ &\quad x = 11 \quad y = 6 \quad z = 7 \end{aligned}$$

$$= 22 e^{42} \cdot 6 + 121 \cdot 7 \cdot e^{42} \cdot 2 + 121 \cdot 6 \cdot e^{42} \cdot 2$$

$$= e^{42} (132 + 121 \cdot 14 + 121 \cdot 12) = \boxed{3278 e^{42}}$$

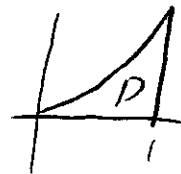
11. (10 points) Find the equation of the plane containing the points $\overset{P}{(1, 2, 3)}, \overset{Q}{(-2, 1, 1)}, \overset{R}{(-3, 1, 2)}$.

$$\vec{PQ} = (-3, -1, -2) \quad \vec{PR} = (-4, -1, -1)$$

$$\vec{PQ} \times \vec{PR} = (-1, 5, -1)$$

$$\boxed{-x + 5y - z = 6}$$

12. (10 points) Let D be the region enclosed by the curves $y = 0$, $y = x^2$ and $x = 1$. Find the average value of $f(x, y) = x \sin y$ over the region D .



$$\begin{aligned} \text{Area } D &= \iint_D 1 = \int_0^1 \int_0^{x^2} 1 \, dy \, dx \\ &= \int_0^1 x^2 \, dx = 1/3 \end{aligned}$$

$$\begin{aligned} \iint_D x \sin y &= \int_0^1 \int_0^{x^2} x \sin y \, dy \, dx \\ &= \int_0^1 -x \cos y \Big|_{y=0}^{y=x^2} \, dx \\ &= - \int_0^1 x \cos(x^2) + x \, dx \\ &= -\frac{1}{2} \sin(x^2) + x^2/2 \Big|_0^1 \\ &= -\frac{1}{2} \sin(1) + 1/2 \end{aligned}$$

$$f_{\text{ave}} = \frac{\iint_D f}{\text{Area}} = \frac{-\frac{1}{2} \sin(1) + 1/2}{1/3}$$

$$= \left(-\frac{3}{2} \sin(1) + \frac{3}{2} \right)$$

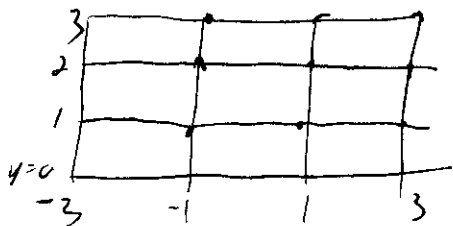
13. (5 points) Consider a particle with position at time t given by $\vec{r}(t) = (1+t, t^3, t^2)$. What is the speed of the particle at when $t = 3$?

$$\vec{r}' = (1, 3t^2, 2t)$$

$$\vec{r}'(3) = (1, 27, 6)$$

$$|\vec{r}'(3)| = \sqrt{1^2 + 27^2 + 36} = \sqrt{764}$$

14. (10 points) Let R be the rectangle $[-3, 3] \times [0, 3]$. Estimate $\iint_R xy^2$ using a Riemann sum with $m = n = 3$ and the upper right corner of each rectangle as your sample point.



Each rect has area 2,

$$2 (f(-1,1) + f(1,1) + f(3,1) + f(-1,2) + f(1,2) + f(3,2) \\ + f(-1,3) + f(1,3) + f(3,3))$$

$$= 2 (-1 + 1 + 3 - 4 + 4 + 12 + -9 + 9 + 27)$$

$$= 2(42) = 84$$