

## Lecture 9

Review Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  and  $\vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ .  
Then for  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  say

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L \quad \text{if}$$

For all  $\epsilon > 0$  there is a  $\delta > 0$  so that

$$\text{If } 0 < |\vec{x} - \vec{a}| < \delta \text{ then } |f(\vec{x}) - L| < \epsilon$$

↑  
magnitude of  
vector  
 $\vec{x} - \vec{a}$

↑  
absolute  
value

Prnk Can generalize even more,  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = \vec{L}$ .

$$\text{Ex } f(x, y, z) = \left( \frac{x^2 + y}{2}, \sin(|xy|) \right)$$

## Properties of Limits

1. If limit =  $L$  then it =  $L$  approaching  $\vec{a}$  along any curve. Thus two curves w/ different limits mean limit DNE.

2. All old limit rules still hold, e.g.

$$\bullet \lim_{\vec{x} \rightarrow \vec{a}} (f(\vec{x}) + g(\vec{x})) = \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) + \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) \quad \text{if both exist}$$

$$\bullet \lim_{\vec{x} \rightarrow \vec{a}} (c f(\vec{x})) = c \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})$$

• Squeeze Thm: If  $g(\vec{x}) \leq h(\vec{x}) \leq f(\vec{x})$  for all  $\vec{x}$  in some ball around  $\vec{a}$  and  $\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) = \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L$

$$\text{Then } \lim_{\vec{x} \rightarrow \vec{a}} h(\vec{x}) = L.$$

Example

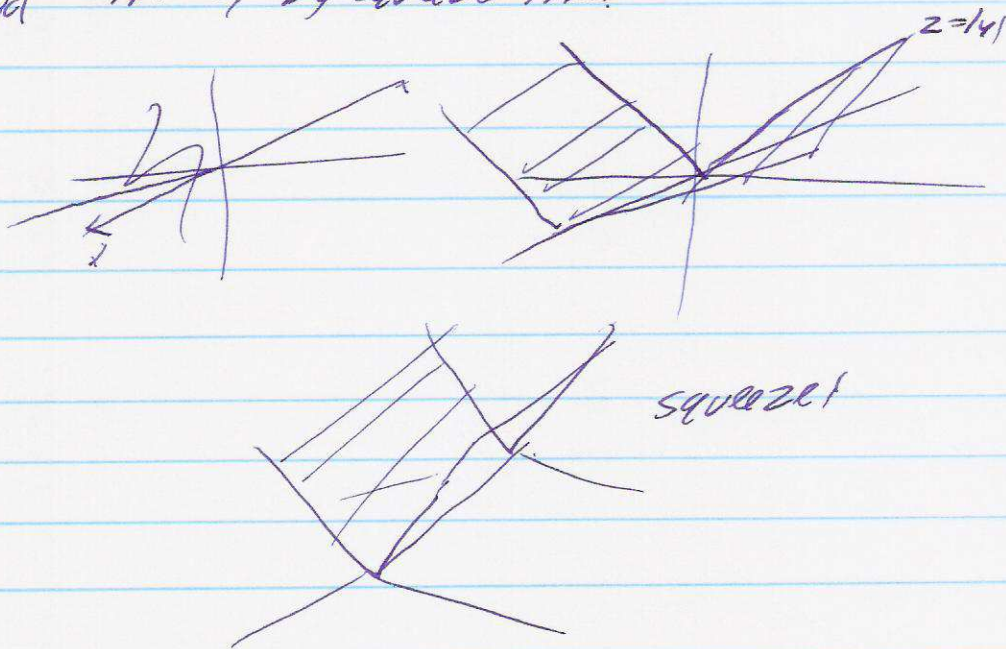
$$f(x,y) = \frac{3x^2y}{x^2+y^2}$$

$$0 < \frac{3x^2}{x^2+y^2} < 3 \quad \text{so}$$

$$0 < \frac{3x^2}{x^2+y^2} |y| < 3|y| \quad \text{Thus}$$

$$-3|y| < f(x,y) < 3|y| \quad \text{and} \quad \lim_{(x,y) \rightarrow (0,0)} 3|y| = 0$$
$$= \lim_{(x,y) \rightarrow (0,0)} -3|y|$$

Thus  $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$ , by Squeeze Thm.



### Review Calc I

Def  $f(x)$  is continuous at  $x=a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

- Reqs
- limit exists and...
  - $f(a)$  is defined and...
  - they are equal /

Limit rules tell us.

Thm Suppose  $f(x)$ ,  $g(x)$  are continuous at  $x=a$ . So are

$$f(x)g(x), f(x)^n, \sqrt[n]{f(x)}, f(x) \pm g(x), c f(x),$$

$$\frac{f(x)}{g(x)} \text{ if } g(a) \neq 0 \text{ etc.,}$$

Example

$$f(x) = \frac{e^x \sin x + x^3}{2x-3} \text{ is continuous except at } x = 3/2$$

CALC III - Same!

Def  $f(\vec{x})$  is continuous at  $\vec{x} = \vec{a}$  if  $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = f(\vec{a})$

Thm As above,

Example

$$\text{Find } \lim_{(x,y,z) \rightarrow (1,2,-1)} \frac{x^2 y}{y+2}$$

Answer  $f(x,y,z)$  is continuous where defined so

$$f(1,2,-1) = \frac{2}{1} = 2 \text{ is the limit.}$$

Example Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin|x,y|}{x^2+y}$

A: Limit = 0

Example

Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy \cos y}{3x^2 + y^2}$

Answer

- Not defined at (0,0) so not continuous

Along  $x=y$   $f(x,y) = \frac{x^2 \cos x}{4x^2} = \frac{\cos x}{4}$  limit = 1

Along  $x=0$   $f(x,y) = 0$  limit = 0.

Thus  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  DNE.

Example

$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$

here we have "0/0"

try to cancel

$= \lim_{(x,y) \rightarrow (0,0)} \frac{y}{\frac{\sqrt{x^2 + y^2}}{x}} = \lim_{(x,y) \rightarrow (0,0)} \frac{y}{\sqrt{1 + \frac{y^2}{x^2}}}$

Notice  $0 < \frac{|y|}{\sqrt{1 + \frac{y^2}{x^2}}} < |y|$  so

$-|y| < f(x,y) < |y|$

so limit = 0 by S.T.

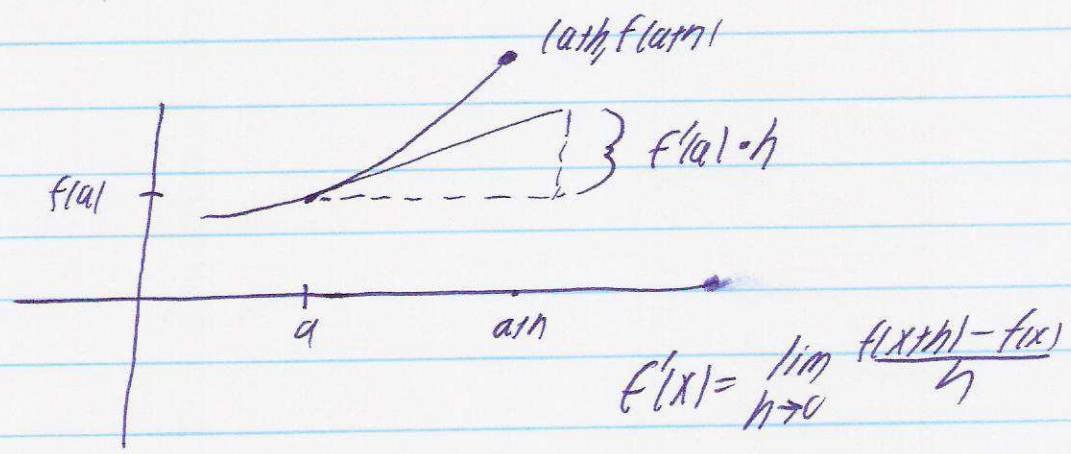
Example

Use polar coordinates to find  $\lim_{(x,y) \rightarrow (0,0)} \frac{e^{-x^2 - y^2} - 1}{x^2 + y^2}$

$= \lim_{(x,y) \rightarrow (0,0)} \frac{e^{-r^2} - 1}{r^2} \stackrel{LHR}{=} \lim_{r \rightarrow 0} \frac{-2re^{-r^2}}{2r}$   
 $= \lim_{r \rightarrow 0} \frac{-2e^{-r^2}}{2} = -1$

# Partial Derivatives

## Review



Change  $x$  from  $a$  to  $a+h$ ,  
 $f(x)$  changes by  $\begin{cases} f(a+h) - f(a) & \text{actual amt} \\ f'(a)h & \text{approximate} \end{cases}$

Suppose  $f(x, y, z)$  and we change only one variable while holding others constant, how does  $f(x, y, z)$  change?

Def  $f_x(x, y)$

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Ex  $f(x, y) = x \cos(xy)$

$$f_x = \cos(xy) + x \cdot (-y \sin(xy))$$

$$f_y = -x^2 \sin(xy)$$