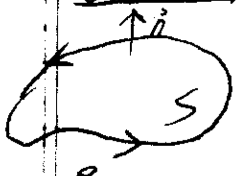
Lecture 24

Review

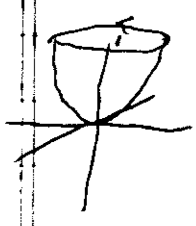
Stokes' Thm $\vec{F}$ vector field on $\mathbb{R}^3$, $\mathcal{C}$ closed curve bounding S



Pos orientation: Walk on curve, head pointing $\vec{n}$
Then surface on left

Then $$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dS$$

Example $\vec{F} = (y^2, x, z^2)$ S = paraboloid $z = x^2+y^2$ below $z=1$, oriented upward



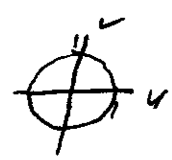
Verify Stokes

LHS $\mathcal{C} = (\cos t, \sin t, 1)$ $\quad 0 \le t \le 2\pi$ $\qquad r'(t) = (-\sin t, \cos t, 0)$

$$\int_0^{2\pi} \vec{F}(r(t))\cdot r'(t)\,dt = \int_0^{2\pi} (\sin^2 t, \cos t, 1)\cdot(-\sin t, \cos t, 0)\,dt$$

$$= \int_0^{2\pi} -\sin^3 t + \cos^2 t\,dt = 0 + \pi = \pi$$

RHS S: $\vec{r}(u,v) = (u, v, u^2+v^2)$ $\qquad (u,v) \in$



$\operatorname{curl}\vec{F} = (0, 0, 1-2y)$ $\qquad r_u = (1,0,2u)$ $\quad r_v = (0,1,2v)$

$r_u \times r_v = (-2u, -2v, 1)$ pts up!

$$\iint_D \operatorname{curl} \vec{F}(\vec{r}(u,v)) \cdot (r_u \times r_v) = \iint_D (0,0,1-2v)\cdot(-2u,-2v,1)$$

$$= \iint_D 1-2v = \int_0^1 \int_0^{2\pi} (1-2r\cos\theta) r \, d\theta \, dr$$

$$= \int_0^1 \int_0^{2\pi} r - 2r^2\cos\theta \, d\theta$$

$$= \int_0^1 r\theta - 2r^2 \sin\theta \Big|_0^{2\pi}$$

$$= \int_0^1 2\pi r = \pi r^2 \Big|_0^1 = \pi$$

They agree!

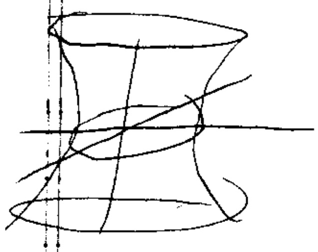
<u>Divergence Thm</u> E simple, solid region in $\mathbb{R}^3$.
S = boundary of E w/ outward pointing normal
$\vec{F}$ = vector field w/ cont partials.

$$\iint_S \vec{F}\cdot\vec{n} = \iiint_E \operatorname{div} \vec{F} \, dV$$

<u>Example</u>

1. Verify for $E = \{x^2+y^2 \le 1, 0 \le z \le 1\}$ $\vec{F} = (xy, yz, zx)$

2. Use Div Thm to calculate Flux of $\vec{F} = (x^3y, -x^2y^2, -x^2yz)$ across S = surface of solid bounded by hyperboloid $x^2+y^2-z^2=1$ and planes $z=-2$, $z=2$



$\text{div}\, F = 3x^2y - 2x^2y - x^2y = 0$ ✓

Flux is zero!!

3. Evaluate $\iint_S \vec{F}\cdot\vec{n}$ for $\vec{F} = (2x, x^2 - xz^2, x^2y - y^3)$

and S is sphere $x^2+y^2+z^2=1$ w/ inward normal!

$\text{div}\, F = 2$ $\quad -\iiint_S 2 = -2$!

4. As above, S is tetrahedron formed by 3 coord planes and $x+y+z=1$ w/ outward normal

$$F = (3x^2+z^2, xy - z^3, z + x^2 - y^2)$$

Stokes Thm

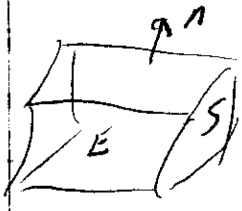
$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} = \oint_C \vec{F} \cdot d\vec{r}$$

Special case:

Green's Thm $\quad C$ closed curve $\quad \vec{F} = (P, Q)$

$$\oint_C \vec{F} \cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

Gauss Thm



$$\iiint_E \operatorname{div} \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n}$$