

Lecture 18 / 19

So far: 1. $f: \mathbb{R} \rightarrow \mathbb{R}^n$ $f(t) = (t, \cos t, t^2 + 1)$ space curves

2. Real valued functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$

EX $f(x, y, z) = x^2 y + y z$

Vector calculus

3. $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ vector valued functions

EX $f(x, y) = (xy, y^2)$

each point is assigned a vector

EX $f: \mathbb{R}^n \rightarrow \mathbb{R}$ then $\nabla f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

for above $\nabla f = (2xy, x^2 + z, y)$

Def When $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ we call \vec{F} a vector field,
each point is assigned a vector, can represent by drawing vectors.

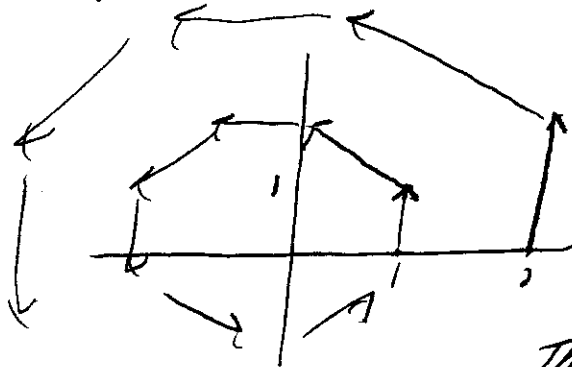
EX $\vec{F}(x, y) = (-y, x)$

$F(0, 0) = (0, 0)$

$F(1, 1) = (-1, 1)$

$F(2, 0) = (0, 2)$

$F(0, 1) = (-1, 0)$

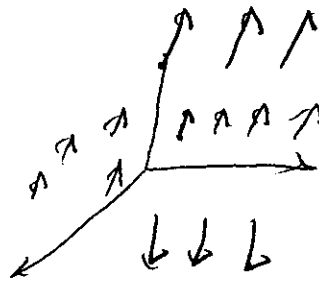


Think: rotating ccwise
faster away from (0,0)

Prop $(x, y) \cdot \vec{F}(x, y) = (x, y) \cdot (-y, x) = 0$

Always \perp

Ex. $F(x, y, z) = 19921$

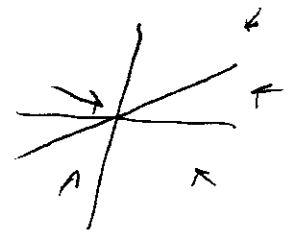


Rank Computer very useful for sketching vector fields!

Ex. $\vec{F}(x) = \frac{-mM_G}{|\vec{x}|^3} \vec{x}$ gravitational force field

$$= \left(\frac{-mM_G x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mM_G y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-mM_G z}{(x^2 + y^2 + z^2)^{3/2}} \right)$$

$|\vec{F}(x)| \rightarrow \infty$ as $\vec{x} \rightarrow 0$



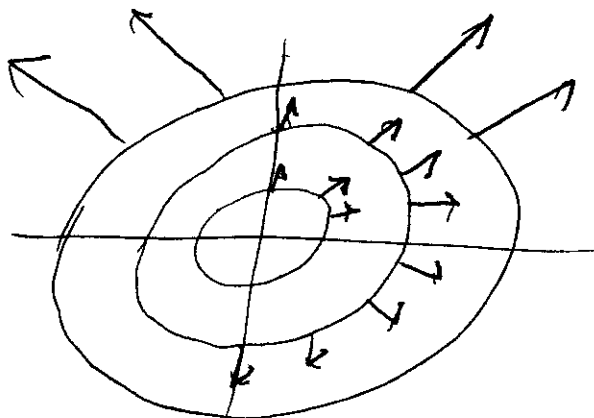
Special Case

Whenever $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a function then ∇f is a vector field

Ex $f(x, y) = x \sin(xy)$ $\nabla f(x, y) = (\sin(xy) + xy \cos(xy), x^2 \cos(xy))$

Recall ∇f is always \perp to level curves of f .

Ex $F(x, y) = 6x^2 + y^2$ $\nabla F = (2x, 2y)$



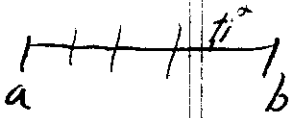
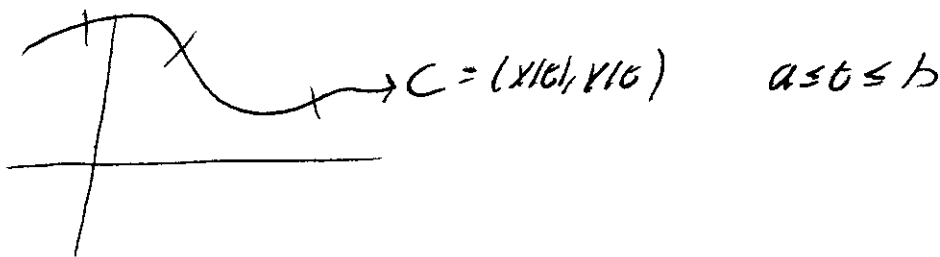
DEF A vector field \vec{F} is conservative if $\vec{F} = \nabla f$ for some f .
 f is called a potential function for \vec{F}

EX $f(x,y,z) = \frac{mMg}{\sqrt{x^2+y^2+z^2}}$ $\nabla f =$ Grav Force field

EX $\vec{F} = (e^{xy} + xye^{xy}, x^2 e^{xy})$ is conservative
with potential function $f(x,y) = xe^{xy}$

Line Integrals

• Goal: Integrate a function "over a curve"



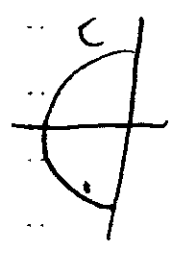
1. Divvy $[a,b]$ into pieces
2. Let $(x_i^*, y_i^*) = (x(t_i^*), y(t_i^*))$
3. Add up

Def $\int_C f(x,y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) ds$

Rmk $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

line integral w.r.t. arc length

Ex Do $\int_C xy^3 ds$ where C is left side of circle of radius 2 ~~at~~ center $(0,0)$



$$x(t) = (2\cos t, 2\sin t) \quad \pi/2 \leq t \leq 3\pi/2$$
$$ds = \sqrt{(-2\sin t)^2 + 2\cos t^2} = 2$$

$$\int_{\pi/2}^{3\pi/2} 2\cos t \cdot 4\sin^3 t \cdot 2 dt = \int_{\pi/2}^{3\pi/2} 16\sin^3 t \cos t dt$$
$$= \frac{16}{3} \sin^3 t \Big|_{\pi/2}^{3\pi/2}$$
$$= \frac{16}{3} (-1 - 1) = \boxed{-32/3}$$

Ex $\int \sin x dx + \cos y dy$ $C =$ top half unit circle from $(1,0)$ to $(-1,0)$ the line segment $(-1,0)$ to $(2,3)$

Line integrals w.r.t. x or y

$$\int_C f(x,y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x,y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

Ex $\int y^2 dx + x dy$ 1. $C =$ arc of parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.
2. $C =$ line segment same points

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Curve in 3 space $\int_C f(x,y,z) ds = \int f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

General Formula

curve $\vec{r}(t)$; $a \leq t \leq b$

$$\int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt = \int_C f(x,y,z) ds$$

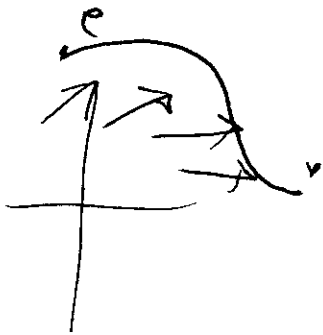
Prob Special case $A.L. = \int_a^b |\vec{r}'(t)| dt$

Ex 1. $\int_C 2x + 5z ds$ $C = (t, t^2, t^3) \quad 0 \leq t \leq 1$

2. $\int_C 2x + 5z dz$

Recall $Work = \vec{F} \cdot \vec{D}$

Problem Calculate work along a curve in a vect field \vec{F}



add up $\vec{F} \cdot \text{unit tangent}$

$$W = \int_C \vec{F} \cdot \vec{T} ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \cdot |\vec{r}'(t)| dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

DEF A \vec{F} a continuous vector field defined on C given by $\vec{r}(t)$ $a \leq t \leq b$ then $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$

Ex. Find work done by $F(x,y) = (x^2, -xy)$ moving a particle along quarter circle from $(1,0)$ to $(0,1)$

Recall $\int_C \vec{F} \cdot d\vec{r} = - \int_{-C} \vec{F} \cdot d\vec{r}$
← opp dir.

Final Observation Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

Then $\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy + R dz$

Problems

p. 1043 # 5, 16, 18, 20, 21, 39, 45