Lecture 24

Review Taylor series of $f(x)$ at $x=a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

- Taylor Remainder Thm For convergence

Ex $f(x) = \frac{1}{1+x}$ Find Maclaurin series from Def.

$f(x) = \frac{1}{1+x}$ $f'(x) = \frac{-1}{(1+x)^2}$ $f''(x) = \frac{2}{(1+x)^3}$ $f'''(x) = \frac{-3\cdot 2}{(1+x)^4}$ $f^{(4)}(x) = \frac{4\cdot 3\cdot 2}{(1+x)^5}$

$f(0)=1$ $f'(0) = -1$ $f''(0) = 2$ $f'''(0) = -3!$ $f^{(n)}(0) = (-1)^n n!$

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n = 1 - x + x^2 - x^3 + x^4 \ldots$$

Mult and Division Taylor series multiply and divide like polynomials

Ex $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$$e^x \cos x = \begin{array}{l} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\ \hline 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \\ \quad - \frac{x^2}{2} \quad - \frac{x^3}{2} \quad - \frac{x^4}{4} \quad - \frac{x^5}{12} \\ \quad \frac{x^4}{24} \quad + \frac{x^5}{24} \ldots \\ \hline 1 + x - \frac{x^3}{3} \ldots \end{array}$$

Ex

$$\tan x = \frac{\sin x}{\cos x}$$

$$\begin{array}{r|l} & x + \tfrac{1}{3}x^3 \\ \hline 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\cdots & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \\ & x - \frac{x^3}{2} + \frac{1}{24}x^5 \\ \hline & \frac{1}{3}x^3 - \frac{1}{30}x^5 \end{array}$$

etc...

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

<u>Fact</u>

- Multiplying two Taylor Series w/ ROC = R gives new one with ROC at least R

- Dividing by $\sum b_n x^n$ w/ $b_0 \neq 0$ gives Taylor series converging for some positive ROC.

Applications

Recall Tangent line to $y=f(x)$ at $(a, f(a))$ is

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$ This is 1^{st} Taylor polynomial!!

Thm $T_n(x)$ gives the best degree n approximation to $f(x)$ in a certain precise sense.

Ex $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ How big is error if $-.3 \le x \le .3$? Approximate $\sin(8^\circ)$ this way.

A $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ so by A.S. error estimate

$$|\text{Error}| \le \frac{.3^7}{7!} = .00000004339$$

$$\sin(8^\circ) = \sin\left(\frac{8\pi}{180}\right) = \sin\left(\frac{2\pi}{45}\right) \approx \frac{2\pi}{45} - \frac{(2\pi/45)^3}{6} + \frac{(2\pi/45)^5}{120}$$

$$= .1396263 4 - .0004536812 + .000000442$$

$$= .1391731 0$$

$$\sin(8^\circ) \text{ actually } = .13917310 0$$

Rmk Such simplifications often used in physics

Indeed for x near 0, $\sin(x) \approx x$, $|\text{error}| \le \frac{x^3}{6}$

Problems

p. 755 # 9, 13, 26

p. 747 # 51, 56