

Lecture 19

Review Series $a_1 + a_2 + \dots = \sum_{n=1}^{\infty} a_n$, take sequence of partial sums:

$$S_n = a_1 + a_2 + \dots + a_n. \quad (\text{Note: } a_n = S_n - S_{n-1}!).$$

Def $\sum_{n=1}^{\infty} a_n = S$ if $\lim_{n \rightarrow \infty} S_n = S$.

Example Geometric Series $a + ar + ar^2 + \dots = \sum_{n=1}^{\infty} ar^{n-1}$
converges to $\frac{a}{1-r}$ if $|r| < 1$, diverges otherwise.

Ex Harmonic series $\sum \frac{1}{n}$ diverges.

Ex/Thm $\sum a_n$ can't possibly converge unless $\lim_{n \rightarrow \infty} a_n = 0$.

Ex $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} + \frac{5}{720}$

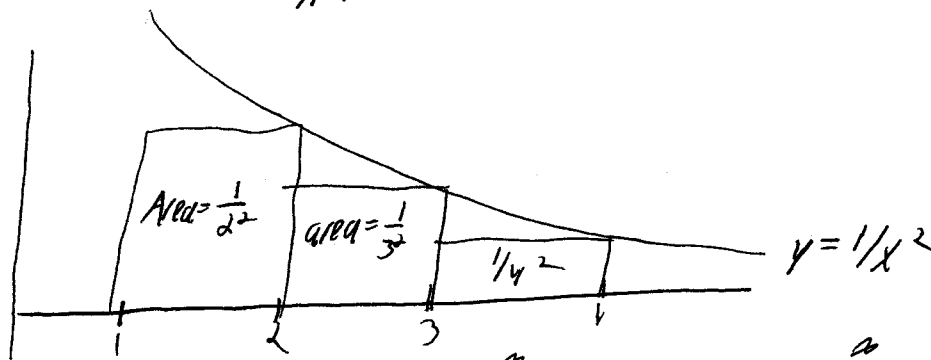
$S_1 = \frac{1}{2}$ $S_2 = \frac{5}{6}$ $S_3 = \frac{23}{24}$ $S_4 = \frac{119}{120}$ Check $S_n = \frac{n!-1}{n!}$ by induction

so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

11.3 Integral Test and Estimates of Sums

Remark Only in special cases (geom series, telescoping series) do we get formula for S_n , need other techniques to show convergence, estimate $\sum a_n$.

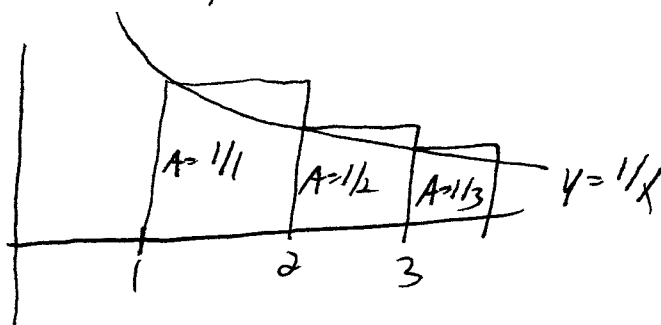
Example I claim $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and is < 2 (actually = $\pi^2/6$)



Clearly $\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_1^{\infty} = 0 - (-1) = 1$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2$.

Example I claim $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.



Clearly $0 < \int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n}$ but $\int_1^{\infty} \frac{1}{x} dx = \infty$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Then (Integral Test) Suppose $f(x)$ is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ converges

Remarks 1. $\sum_{n=4}^{\infty} f(n)$ etc. - works, e.g. $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$
converges $\leftrightarrow \int_4^{\infty} \frac{1}{(x-3)^2} dx$ does.

2. Only need $f(n)$ eventually decreasing, since changing beginning terms of series doesn't change convergence.

Ex. Does $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ converge?
 $\frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots$

$f(x) = \frac{1}{x^2+1}$ is continuous, decreasing and $\int_2^{\infty} \frac{1}{x^2+1} dx = \tan^{-1} x \Big|_2^{\infty} = \frac{\pi}{2} - \tan^{-1} 2$

YES.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.
called p-series.

$p=3$, $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \dots$ converges

OPEN PROBLEM: TO WHAT? (open for odd)

EX. Show $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges

Proof $f(x) = \frac{\ln x}{x}$ $f'(x) = \frac{1-\ln x}{x^2}$ is < 0 for $x > e$
so decreasing eventually

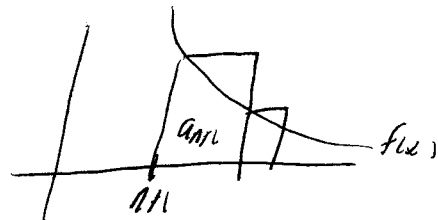
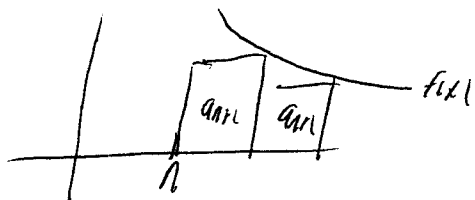
$\int_1^{\infty} \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 \Big|_1^{\infty}$ DIVERGES

Remainder Estimate Suppose f continuous, positive, decreasing for $x \geq 1$

Suppose $\sum a_n = S$ converges Then:

$$\text{If } R_n = S - S_n \text{ then } \sum_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

Proof $R_n = a_{n+1} + a_{n+2} + \dots$

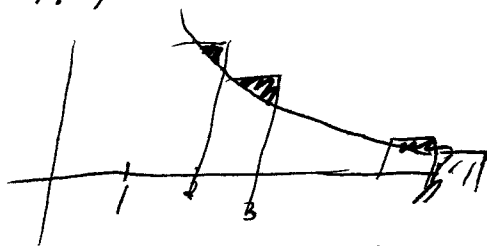


Problems

- Find S_{10} for series $\sum_{n=1}^{\infty} 1/n^4$, estimate error
Find value of n so S_n is within 0.00001

$$\text{Rmk: } \sum_{n=1}^{\infty} 1/n^4 < 1/40$$

- Let $t_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$



$t_n = \text{red area}$

Notice that $t_n > 0$

Check $t_n - t_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1}$, show > 0 (disturbed)

OR t_n is mon-decreasing, bounded below, converges to

$V = \text{Euler Constant}$

OPEN PROBLEM: IS V rational?

IF SO, DENOM IS $> 10^{24280}$

Comparison Tests Suppose $\sum a_n, \sum b_n$ series w/ positive terms.

1. If $a_n \leq b_n$ for all n and $\sum b_n$ converges then $\sum a_n$ converges.
2. If $b_n \leq a_n$ for all n and $\sum b_n$ diverges then $\sum a_n$ diverges.

Use to compare against:

- p series $\sum 1/n^p$ converges if & only if $p > 1$
- geom $\sum ar^n$ convs if & only if $|r| < 1$

EX

1. Show $\sum_{n=1}^{\infty} \frac{6}{2^n + 1}$ converges

2. Show $\sum_{n=1}^{\infty} \frac{\ln(n)^3}{n}$ diverges

Limit Comparison Test Suppose $\sum a_n, \sum b_n$ series w/ positive terms

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ then both converge or both diverge.

EX Does $\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n^2 + 1}$ converge?

EX Does $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$ converge?