

Name: SOLUTIONS

Math 142- Midterm Exam #3 - April 23, 2009

1. (10 points) Complete the following definitions precisely:

a. A sequence $\{a_n\}$ has the **limit** L , and we write $\lim_{n \rightarrow \infty} a_n = L$ if \dots .

For any $\epsilon > 0$ there is an N
such that $|a_n - L| < \epsilon$ whenever $n > N$.

b. Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

we say the series is **convergent** if \dots .

the sequence of partial sums

$\{s_n\}$ converges

where

$$s_n = a_1 + a_2 + \dots + a_n$$

2. (10 points) Decide if the following series converge. If they do, then evaluate the sum.

a.

$$3 - 3/2 + 3/4 - 3/8 + 3/16 - 3/32 + \dots$$

Geometric $a = 3$ $r = -1/2$

$$\frac{3}{1 - (-1/2)} = \frac{3}{\frac{3}{2}} = \textcircled{2}$$

b.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$n=0 \Rightarrow A = 1/2$$

$$n=-2 \Rightarrow B = -1/2$$

$$\sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2}$$

$$= \frac{1}{2} (1 - 1/3 + 1/2 - 1/4 + 1/3 - 1/5 + 1/4 - 1/6 + 1/5 - 1/7 + \dots)$$

telescoping

$$= \frac{1}{2} (1 + 1/2) = \textcircled{3/4}$$

3. (30 points) Decide whether the series converges absolutely, converges conditionally, or diverges. Justify your answer with the appropriate tests.

a.

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2} = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0$$

converges absolutely
by ratio test

b.

$$\sum_{n=1}^{\infty} \frac{2}{\pi^n + 3}$$

$$0 < \frac{2}{\pi^n + 3} < \frac{2}{\pi^n}$$

and $\sum \frac{2}{\pi^n}$ converges (geometric, $r = 1/\pi$)

Thus this converges absolutely
by comparison
test

c.

$$\sum_{n=2}^{\infty} \frac{n^5}{2^n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^5}{2n^5} = \frac{1}{2} \end{aligned}$$

converges absolutely
by ratio test

d.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{6n+15}$$

converges by A.S.T.

But not absolutely by L.C.T. to $\leq 1/2$

Thus it

converges conditionally

e.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

Diverges by L.C.T.
with $\sum \frac{1}{n}$

f.

$$1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(2n-1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \dots 2n-1 \cdot 2n+1}{(2n+1)!} \cdot \frac{(2n-1)!}{1 \cdot 3 \cdot 5 \dots 2n-1}$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{(2n+1)(2n)} = 0$$

converges abs
by ratio test

4. (15 points) True or false:

- F a. Every bounded sequence converges.
T b. If $\sum a_n$ converges then $\lim_{n \rightarrow \infty} a_n = 0$.
T c. If $\sum c_n x^n$ diverges for $x = 6$ then it diverges for $x = 10$.
F d. The Ratio Test can be used to determine if $\sum 1/n^3$ converges.
T e. $1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$ diverges.
F f. If $0 \leq a_n \leq b_n$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

5. (5 points) Let $f(x) = 1/x^2$. Find the Taylor polynomial $T_3(x)$ centered at $a = 2$.

$$f(2) = 1/4$$

$$f'(x) = \frac{-2}{x^3} \quad f'(2) = -1/4$$

$$f''(x) = \frac{6}{x^4} \quad f''(2) = 3/8$$

$$f'''(x) = \frac{-24}{x^5} \quad f'''(2) = \frac{-24}{32} = -3/4$$

$$1/4 + -1/4(x-2) + \frac{3/8}{2!} (x-2)^2 - \frac{3/4}{3!} (x-2)^3$$

$$= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3$$

6. (10 points) Find the interval of convergence:

$$\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n+3}}$$

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{2^{n+1} |x-3|^{n+1}}{\sqrt{n+4}} \cdot \frac{\sqrt{n+3}}{2^n |x-3|^n} \right| \\ &= 2|x-3| \frac{\sqrt{n+3}}{\sqrt{n+4}} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2|x-3| \quad \text{so} \quad \begin{aligned} -1 < 2x-6 < 1 \\ 5 < 2x < 7 \\ 5/2 < x < 7/2 \end{aligned}$$

$$x = 5/2 \quad \sum_{n=0}^{\infty} \frac{2^n (-1/2)^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$$

conv by AST

$$x = 7/2 \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}} \text{ div}$$

$$\left[\frac{5}{2}, \frac{7}{2} \right)$$

7. (10 points) Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (from the definition) or known series.

a. $f(x) = \frac{x^2}{1+x}$.

b. $f(x) = xe^{2x}$.

a. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \dots$

$$\frac{x^2}{1+x} = x^2 - x^3 + x^4 - x^5 + x^6 - \dots \quad R=1$$

b. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad R=\infty$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

$$xe^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1} \quad R=\infty$$

8. (10 points) Use series to approximate:

$$\int_0^1 x \cos(x^3) dx$$

with an error at most $1/1000$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots$$

$$x \cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots$$

$$\int_0^1 x \cos(x^3) = \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{16} + \frac{1}{336} - \frac{1}{14400}$$

Thus

$$\frac{1}{2} - \frac{1}{16} + \frac{1}{336}$$

has $|\text{error}| < \frac{1}{14400}$

$$\begin{array}{r} 24 \\ 14 \\ \hline 90 \\ 270 \\ \hline 336 \end{array}$$