

Name: SOLUTIONS

Math 142- Final Exam - May 7, 2009

Instructions: The exam is worth 150 points. Calculators are not permitted. You are permitted two 3×5 notecards.

1. (10 points) Compute the following integrals.

a. $\int \frac{1}{x^2+9} dx$.

$$\frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C$$

b. $\int \frac{1}{x^2-9} dx$.

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x-3)$$

$$x=3 \rightarrow A = 1/6$$

$$x=-3 \rightarrow B = -1/6$$

$$\int \frac{1/6}{x-3} - \frac{1/6}{x+3} dx$$

$$= \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + C$$

2. (10 points) Compute the following integral.

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$$

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$x^2 \text{ coef: } 2 = A + B$$

$$x \text{ coef: } -1 = C$$

$$\text{const: } 4 = 4A$$

$$A = 1 \quad B = 1 \quad C = -1$$

$$\int \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4} dx$$

$$= \ln|x| + \frac{1}{2} \ln|x^2 + 4| - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

3. (10 points) Compute the following integrals.

a. $\int t \cos(t^2 + 5) dt$.

$$u = t^2 + 5 \quad du = 2t dt$$

$$\int \frac{1}{2} \cos u du = \frac{1}{2} \sin u + C$$

$$= \frac{1}{2} \sin(t^2 + 5) + C$$

b. $\int t \cos(t + 5) dt$.

$$u = t \quad dv = \cos(t + 5) dt$$

$$du = dt \quad v = \sin(t + 5)$$

$$\int u dv = uv - \int v du$$

$$= t \sin(t + 5) - \int \sin(t + 5) dt$$

$$= t \sin(t + 5) + \cos(t + 5) + C$$

4. (10 points) Compute the following integrals.

a. $\int \sec^3 x \tan^3 x dx$. use $\tan^2 x + 1 = \sec^2 x$

$$= \int \sec^2 x \tan^2 x \sec x \tan x dx$$

$$= \int \sec^2 x (\sec^2 x - 1) \sec x \tan x dx \quad u = \sec x \quad du = \sec x \tan x dx$$

$$= \int u^2 (u^2 - 1) du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

b. $\int \frac{\sqrt{9-x^2}}{x^2} dx$.

$$x = 3 \sin \theta \quad dx = 3 \cos \theta d\theta$$

$$\int \frac{\sqrt{9-9\sin^2 \theta}}{9 \sin^2 \theta} \cdot 3 \cos \theta d\theta = \int \frac{3 \cos \theta \cdot 3 \cos \theta}{9 \sin^2 \theta} d\theta$$

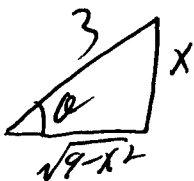
$$= \int \cot^2 \theta d\theta$$

$$= \int (\csc^2 \theta - 1) d\theta$$

$$= -\cot \theta - \theta + C$$

now ~~we~~ ~~find~~

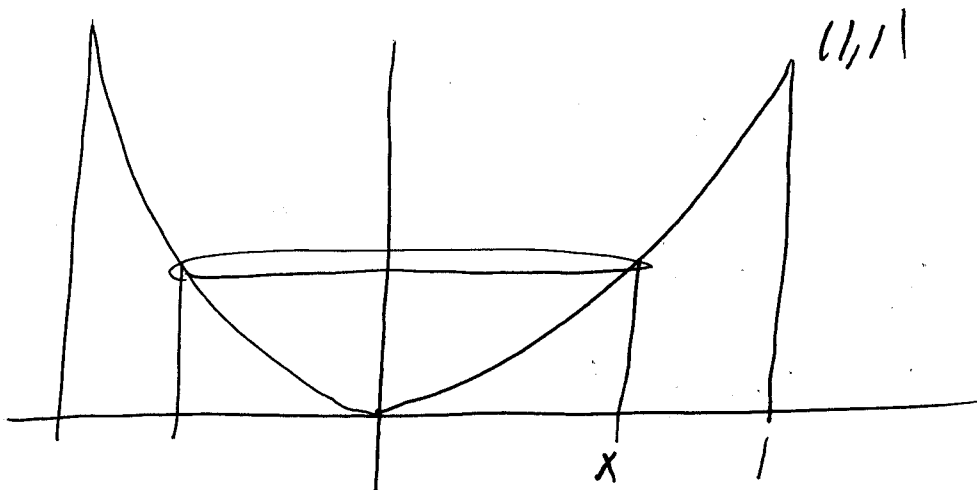
$$\cot \theta = \frac{\sqrt{9-x^2}}{x} \quad \theta = \sin^{-1}\left(\frac{x}{3}\right)$$



Answer:

$$\frac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

5. (10 points) Consider the region bounded by the curves $y = x^2$, $y = 0$ and $x = 1$. Use the method of cylindrical shells to find the volume obtained by rotating this region about the y -axis. Sketch the region and a typical shell.



$$\text{radius} = x$$

$$\text{height} = x^2$$

$$V = \int 2\pi r h \, dr = \int_0^1 2\pi x \cdot x^2 \, dx$$

$$= \int_0^1 2\pi x^3 \, dx = \frac{\pi x^4}{2} \Big|_0^1$$

$$= \pi/2$$

6. (10 points) Evaluate the following integrals.

a. $\int_0^9 \frac{1}{\sqrt[3]{x-1}} dx$.

$$\begin{aligned} &= \int_0^1 (x-1)^{-1/3} dx + \int_1^9 (x-1)^{-1/3} dx \\ &= \frac{3}{2} (x-1)^{2/3} \Big|_0^1 + \frac{3}{2} (x-1)^{2/3} \Big|_1^9 \\ &= \frac{3}{2} (0-1) + \frac{3}{2} (4-0) \\ &= -\frac{3}{2} + \frac{12}{2} = \frac{9}{2} \end{aligned}$$

must split into
two pieces
for credit.

b. $\int_{-\infty}^0 x e^x dx$.

$$\begin{aligned} u &= x & v &= e^x \\ du &= dx & dv &= e^x dx \end{aligned}$$

$$= x e^x - \int e^x dx$$

$$= \lim_{m \rightarrow -\infty} (x e^x - e^x) \Big|_m^0$$

$$= \lim_{m \rightarrow -\infty} ((-1) - (m e^m - e^m))$$

$$= -1$$

7. (10 points) Let $x = \frac{t}{1+t}$, $y = \ln(1+t)$ for $0 \leq t \leq 2$. ~~Find the exact length of this curve.~~ ^{Set up integral}

$$x' = \frac{1+t-t}{(1+t)^2} \quad y' = \frac{1}{1+t}$$

$$AL = \int_0^2 \sqrt{\frac{1}{(1+t)^4} + \frac{1}{(1+t)^2}} dt$$

8. (10 points) Sketch neatly the polar curve $r = -3 \cos(\theta)$.

$$r^2 = -3r \cos \theta$$

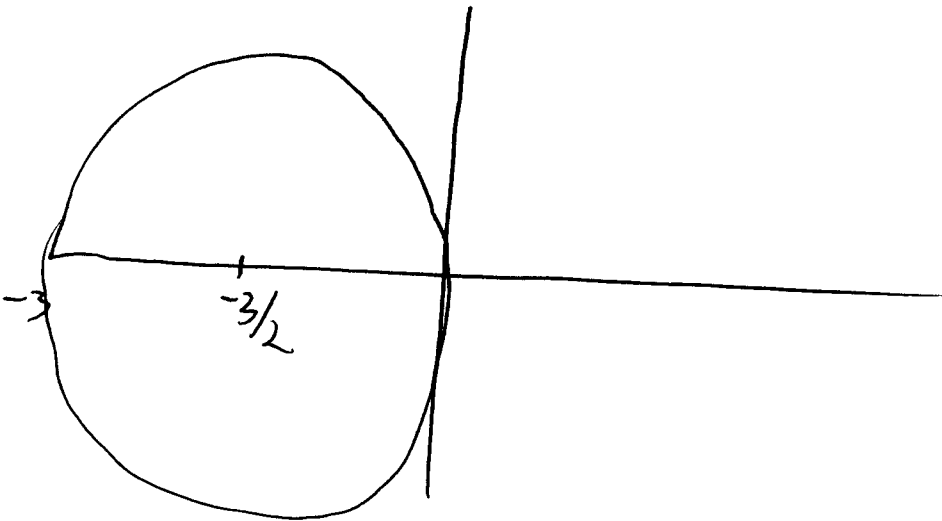
$$x^2 + y^2 = -3x$$

$$x^2 + 3x + y^2 = 0$$

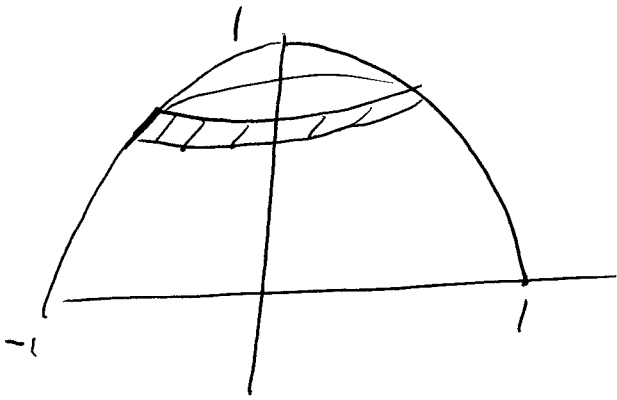
$$x^2 + 3x + \frac{9}{4} + y^2 = \frac{9}{4}$$

$$\cancel{x^2} \quad (x + \frac{3}{2})^2 + y^2 = (\frac{3}{2})^2$$

circle, center $(-\frac{3}{2}, 0)$ radius $\frac{3}{2}$



9. (10 points) Consider the curve $y = 1 - x^2$ for $0 \leq x \leq 1$. Find the surface area obtained by rotating this curve about the y -axis.



integrate $S.A. = \int 2\pi x ds$

$$ds = \sqrt{1 + (-2x)^2} = \sqrt{1 + 4x^2}$$

$$S.A. = \int_0^1 2\pi x \sqrt{1 + 4x^2} dx$$

$$u = 1 + 4x^2 \quad du = 8x dx$$

$$= \int_1^5 \frac{\pi}{4} \sqrt{u} du = \frac{\pi}{6} u^{3/2} \Big|_1^5$$

$$= \frac{\pi}{6} (5^{3/2} - 1)$$

10. (10 points) Consider the parameterized curve $x = e^t + 2$, $y = t^2 + t + 1$. Find the equation of the tangent line to this curve at the point (3, 1).

$$x' = e^t \quad y' = 2t + 1$$

(3, 1) occurs when $t = 0$

$$\frac{dy}{dx} = \frac{\cancel{2t+1}}{\cancel{2t+1} e^t} \quad \text{slope} = \frac{dy}{dx} \Big|_{t=0} = \frac{1}{1} = 1$$

$$y - 1 = 1(x - 3)$$

$$y = x - 2$$

11. (10 points) Complete the following definitions precisely:

a. A sequence $\{a_n\}$ has the **limit** L , and we write $\lim_{n \rightarrow \infty} a_n = L$ if \dots .

For any $\epsilon > 0$ there is $N > 0$ such that

$$|a_n - L| < \epsilon \text{ whenever } n > N.$$

b. Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

we say the series is **convergent** if \dots .

The sequence of partial

sums converges

12. (10 points) Compute $\sin(1)$ and $1/e$ such that the absolute value of the error in each case is $< 1/1000$. Hint: Use series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \dots$$

$\approx < 1/1000$

$$1 - \frac{1}{6} + \frac{1}{120}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots$$

$$e^{-1} \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}$$

13. (10 points) Find the Maclaurin series for

$$f(x) = \frac{x}{x^2 + 16} = \frac{1}{16} \frac{x}{\frac{x^2}{16} + 1}$$

$$= \frac{x}{16} \cdot \frac{1}{1 + (\frac{x}{4})^2}$$

Now

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{1}{1+(\frac{x}{4})^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^{2n}}$$

$$\frac{x}{16} \cdot \frac{1}{1+(\frac{x}{4})^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{4^{2n+2}}$$

$$= \cancel{\frac{x}{16}} - \frac{x^3}{16} + \frac{x^5}{256} - \dots$$

$$\frac{x}{4^2} - \frac{x^3}{4^4} + \frac{x^5}{4^6} - \frac{x^7}{4^8} + \dots$$

14. (10 points) a. Consider

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$$

Determine if this series diverges, converges conditionally or converges absolutely. Justify your answer.

Converges by A.S.T.

However $\sum \frac{n}{\sqrt{n^3+2}}$ diverges by L.C.T w/ $\sum \frac{1}{\sqrt{n}}$

Thus converges conditionally

b. Do the same for the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$.

$$0 \leq \left| \frac{\cos n}{n^2+1} \right| \leq \frac{1}{n^2+1}$$

Thus $\sum \frac{\cos n}{n^2+1}$ converges absolutely

by comparison
test.

15. (10 points)

a. What is a geometric series? Under what circumstances is it convergent? What is its sum?

Geometric Series is of form $a + ar + ar^2 + \dots = \sum_{n=0}^{\infty} ar^n$

Converges if & only if $-1 < r < 1$

If so, then sum is $\frac{a}{1-r}$.

b. Suppose $\sum_{n=0}^{\infty} a_n = 3$ and s_n is the n -th partial sum of the series. What is $\lim_{n \rightarrow \infty} a_n$?
What is $\lim_{n \rightarrow \infty} s_n$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$\lim_{n \rightarrow \infty} s_n = 3$$

