Lecture 3

Review
We are learning to calculate \( \lim_{x \to a} f(x) \), the limit as \( x \) approaches \( a \) of \( f(x) \).

Properties/Informal Def

1. \( f(a) \) is irrelevant to \( \lim_{x \to a} f(x) \), only function near but not at \( x = a \) matters.

2. Limit is determined completely by \( f(x) \) for 
   \( a - \epsilon < x < a + \epsilon \) for any \( \epsilon > 0 \), limit is a "local" property.

3. Informally \( \lim_{x \to a} f(x) = L \) if we can make \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) on either side.

Example  Guess value of \( \lim_{x \to 1} \frac{x^2-1}{x^2+3x+2} \)

Notice \( f(1) \) undefined.

\[
\begin{array}{c|c|c|c|c}
 x & f(x) \\
 -1.1 & 2.333 \\
 -1.01 & 2.0303 \\
 -0.99 & 1.97 \\
 -0.9 & 1.7 \\
\end{array}
\]

6. Guess \( \lim_{x \to 1} \frac{x^2-1}{x^2+3x+2} = -2 \)
\[
\lim_{x \to 0} \frac{x}{\sin x} = 1 \quad \text{(Fundamental Limit)}
\]

1. \( \frac{x}{\sin x} \) at \( x = 0 \) is undefined.
2. Be sure to use radians!

\[\lim_{x \to 0} \frac{x}{\sin x} = 0 \quad \text{so far} \]

**Example:**

\[
\lim_{x \to 0} \frac{x}{\sin x}
\]

**Calculating (Using) L'Hôpital's Rule:**

If \( f(x) = \frac{g(x)}{h(x)} \) and \( f(x) \) and \( h(x) \) are differentiable in an open interval containing \( a \) (except possibly at \( a \)) and \( h'(x) \neq 0 \) in this interval, and if \( \lim_{x \to a} g(x) = 0 \) and \( \lim_{x \to a} h(x) = 0 \) or \( \infty \), then

\[\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)} \]

**Guess:** \( f(x) = \frac{6 - x}{(x + 3)(x + 4) - 3} \)

**Example:**

\[\lim_{x \to -1} \frac{x}{x - 1} \]

**Conclusion:**

If \( f(x) = \frac{g(x)}{h(x)} \) and \( f(x) \) and \( h(x) \) are differentiable in an open interval containing \( a \) (except possibly at \( a \)) and \( h'(x) \neq 0 \) in this interval, and if \( \lim_{x \to a} g(x) = 0 \) and \( \lim_{x \to a} h(x) = 0 \) or \( \infty \), then

\[\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)} \]

**Note:**

\[\lim_{x \to -1} \frac{x - 1}{(x + 1)(x - 1)} = \lim_{x \to -1} \frac{x^2 + x}{(x + 1)} \]
A warning example

Let \( f(x) = \sin(\frac{\pi}{x}) \) defined for \( x \neq 0 \).

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
\frac{1}{2} & 0 \\
\frac{1}{4} & 0 \\
\frac{1}{6} & 0 \\
\frac{1}{8} & 0 \\
\frac{1}{16} & 0 \\
\hline
\end{array}
\]

Looks like \( \lim_{x \to 0} \sin(\frac{\pi}{x}) = 0 \)!

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
2 & 1 \\
\frac{2}{5} & 1 \\
\frac{2}{65} & 1 \\
\frac{2}{13} & 1 \\
\frac{2}{17} & 1 \\
\hline
\end{array}
\]

Looks like \( \lim_{x \to 0} \sin(\frac{\pi}{x}) = 1 \)!!

As \( x \to 0 \), \( \frac{\pi}{x} \to \infty \), so \( \sin(\frac{\pi}{x}) \) goes up and down infinitely often.

\[ y = \sin(\frac{\pi}{x}) \]

\[ \lim_{x \to 0} \sin(\frac{\pi}{x}) \text{ DNE} \]

\[ \text{No Hot Spikes!} \]
Example

\[ f(x) = \begin{cases} 
3 - x & x < 0 \\
-x^2 & x \geq 0 
\end{cases} \]

Notice \( \lim_{x \to 0} f(x) \) DNE. We say

\[ \lim_{x \to 0^+} f(x) = 0 \quad \lim_{x \to 0^-} f(x) = 3 \]

Again \( f(0) \) is irrelevant!

Fact \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^-} f(x) = L \) and \( \lim_{x \to a^+} f(x) = L \).