OPTIMIZATION

Problem: These are just max/min word problems
1. Get equation for what is being optimized
   - sketch
   - eliminate variables?
   - constraints (domain)
2. Use calculus

Example: I have 100 feet of fence to make rectangular pen against a wall. What is largest area I can enclose?

\[ A = xy \]  but
\[ 2y + x = 100 \]
\[ \text{so } x = 100 - 2y \]

\[ A(x) = (100 - 2y)y = 100y - 2y^2 \quad 0 \leq y \leq 50 \]
\[ A'(y) = 100 - 4y \]
\[ 100 - 4y = 0 \]
\[ y = 25 \]

\begin{tabular}{c|c|c|}
\hline
y & A(y) & dy/dy \\
\hline
0 & 0 & \\
25 & 125 & \\
50 & 0 & \\
\hline
\end{tabular}

max area is 1250 sq feet
Ex Find a positive # s.t. sum of number and its reciprocal is as small as possible.

Minimize \( S(x) = x + \frac{1}{x} \) \( w/ \ x > 0 \)

\[ S'(x) = 1 - \frac{1}{x^2} \]

\( 0 = 1 - \frac{1}{x^2} \quad x = \pm 1 \]

\[ \frac{1}{x^2} \quad \text{global min} \]

This \( S(x) \) has a global min at \( x = 1 \).

Ex The top and bottom margins of a poster are each 6 cm, the side margins are 4 cm. If the area of printed material is fixed at 384 cm², find dimensions of poster with smallest area.

Given \( xy = 384 \)

Minimize \((x+8)(y+12)\]

Minimize \( A(x) = (x+8)\left(\frac{384}{x} + 12\right) \]

\[ = 384 + 12x + \frac{384 	imes 12}{x} + 86 \]

\[ A(x) = 480 + 12x + \frac{3072}{x} \]

\[ A'(x) = 12 - \frac{3072}{x^2} \quad \text{set } A'(x) = 0, \ x = 16 \]

so \( y = 24 \)

Answer \( 24 \times 36 \)
Find point on the line $6x + 6y = 9$ closest to $(-3, 1)$

Minimize $D = \sqrt{(x+3)^2 + (y-1)^2}$ with $y = -6x + 9$

$$D(x) = \sqrt{(x+3)^2 + (-6x+9)^2}$$

**Trick:** Minimize $D^2 = (x+3)^2 + (-6x+9)^2$.

$$D^2(x) = 2(x+3) - 12(6x+9)$$
$$= 20x - 54$$
$$0 = 20x - 54$$
$$x = \frac{9}{5}$$

$$y = -6\left(\frac{9}{5}\right) + 9$$
$$y = -\frac{45}{5} + 9$$
$$y = \frac{45}{5} = 9$$

**Answer:** $\left(\frac{9}{5}, \frac{45}{5}, 9\right)$
4. Find largest rectangle with base on x-axis under $y = 8 - 3x^2$.

Maximise $A(x) = 2x(8 - 3x^2)$ \(0 \leq x \leq \frac{\sqrt{10}}{3}\)

$A'(x) = 16 - 18x^2$

$x = \pm \sqrt{\frac{16}{18}} = \pm \frac{4}{3\sqrt{2}}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$A(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{4}{3\sqrt{2}}$</td>
<td>+</td>
</tr>
<tr>
<td>$\sqrt{\frac{10}{3}}$</td>
<td>0</td>
</tr>
</tbody>
</table>

$2 \cdot \frac{4}{3\sqrt{2}} \times 8 - 3 \cdot \frac{16}{10}$

Dimensions $\frac{8}{3\sqrt{2}} \times \frac{9\sqrt{10}}{10}$
Problem

A lifeguard is 10 m from water. A swimmer is 20 m from shore and 40 m sideways from lifeguard. She runs 5 m/sec, swims 2 m/sec. Minimize time to get to swimmer.

\[ T(x) = \frac{\sqrt{100 + (40-x)^2}}{5} + \frac{\sqrt{x^2 + 400}}{2} \]

\[ T'(x) = \frac{1}{5} \cdot \frac{-2(40-x)}{\sqrt{100 + (40-x)^2}} + \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2 + 400}} \]

\[ = \frac{-1}{5} \cos \theta + \frac{1}{2} \cos \phi \]

So if \( T'(x) = 0 \) then

\[ \frac{\cos \phi}{\cos \theta} = \frac{2}{3} \]

(SNELL’S LAW)